SIMP paradigm and hidden gauge theories

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Ref) HML, M.-S. Seo, 1504.00745 [hep-ph];
S.-M. Choi, HML, 1505.00960 [hep-ph].

CosKASI Dark Matter Workshop 2015
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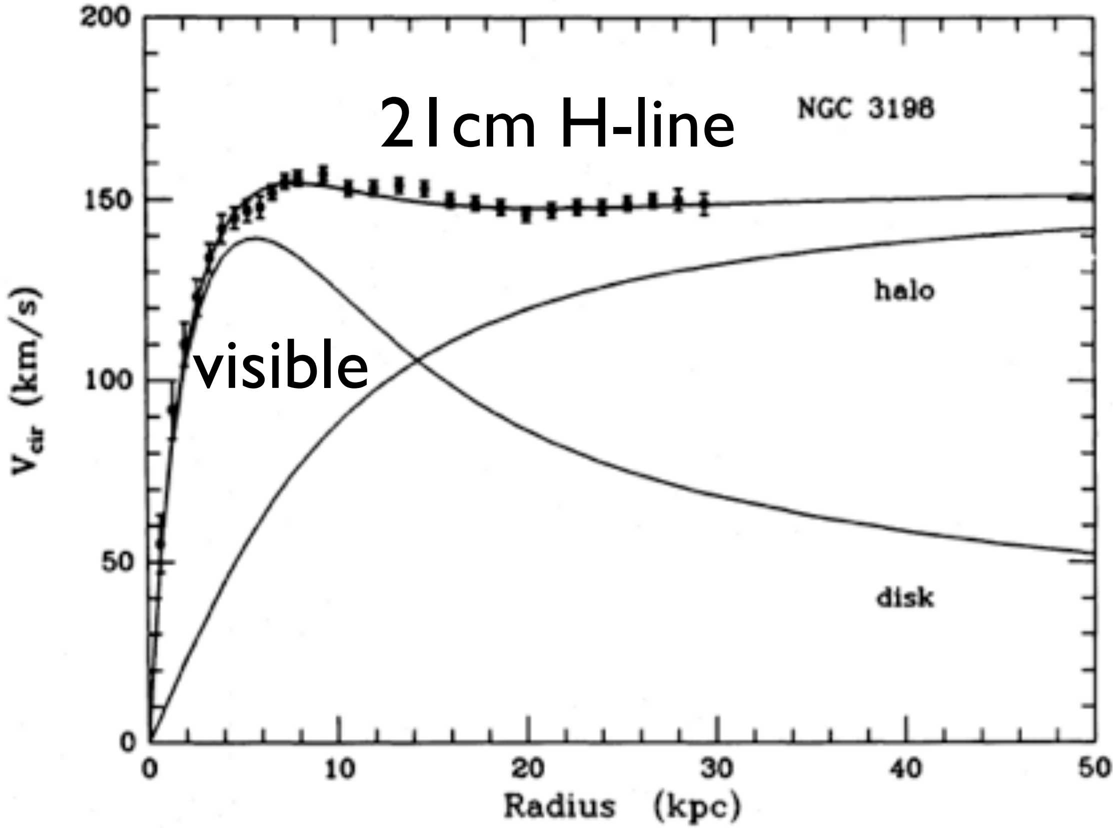
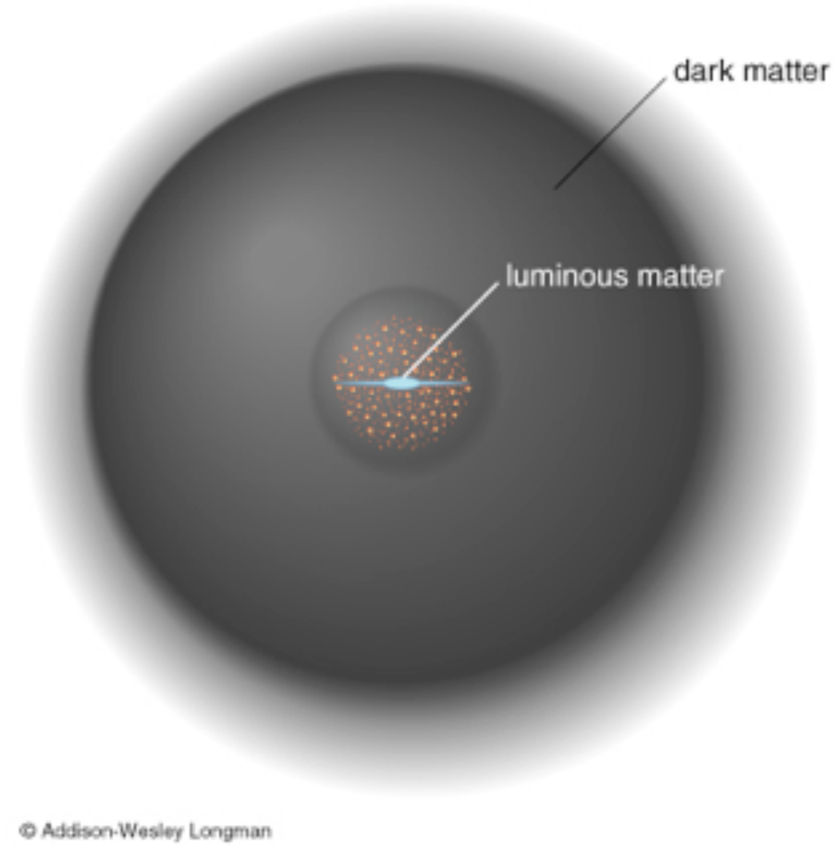
Outline

- SIMP paradigm
- SIMP DM from hidden QCD
- SIMP DM from Z3 symmetry
- Conclusions

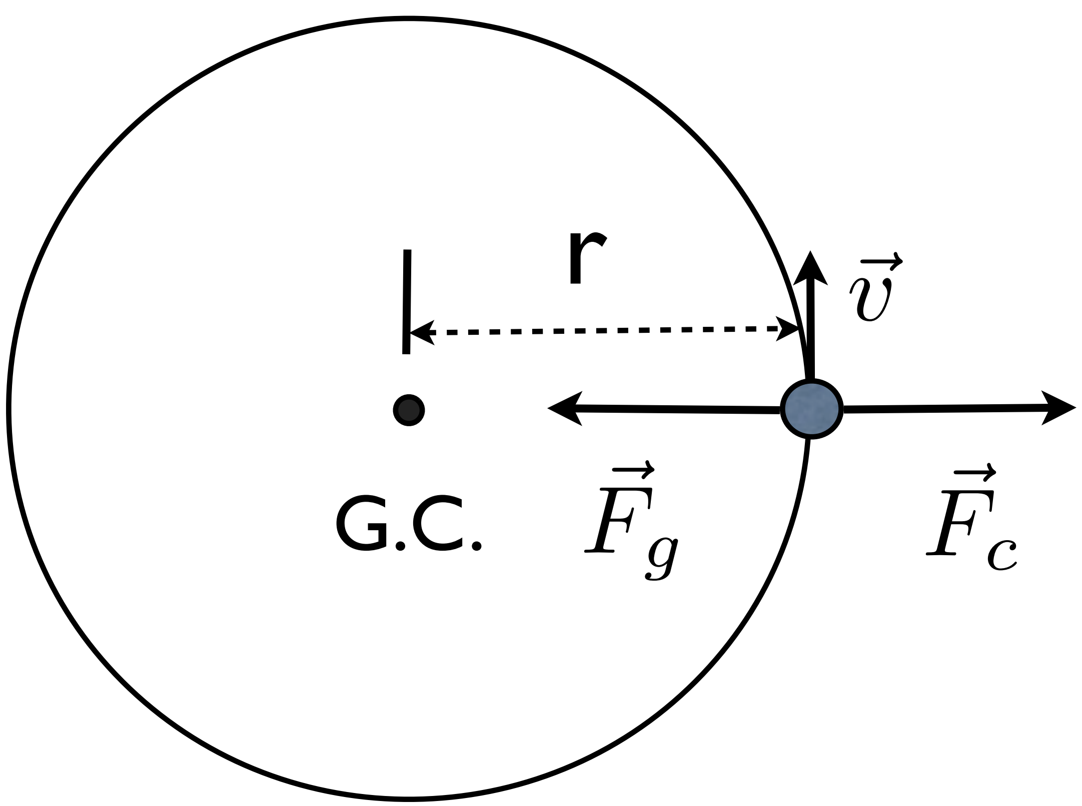
SIMP paradigm

Galaxy rotation curves





F. Zwicky(1933)



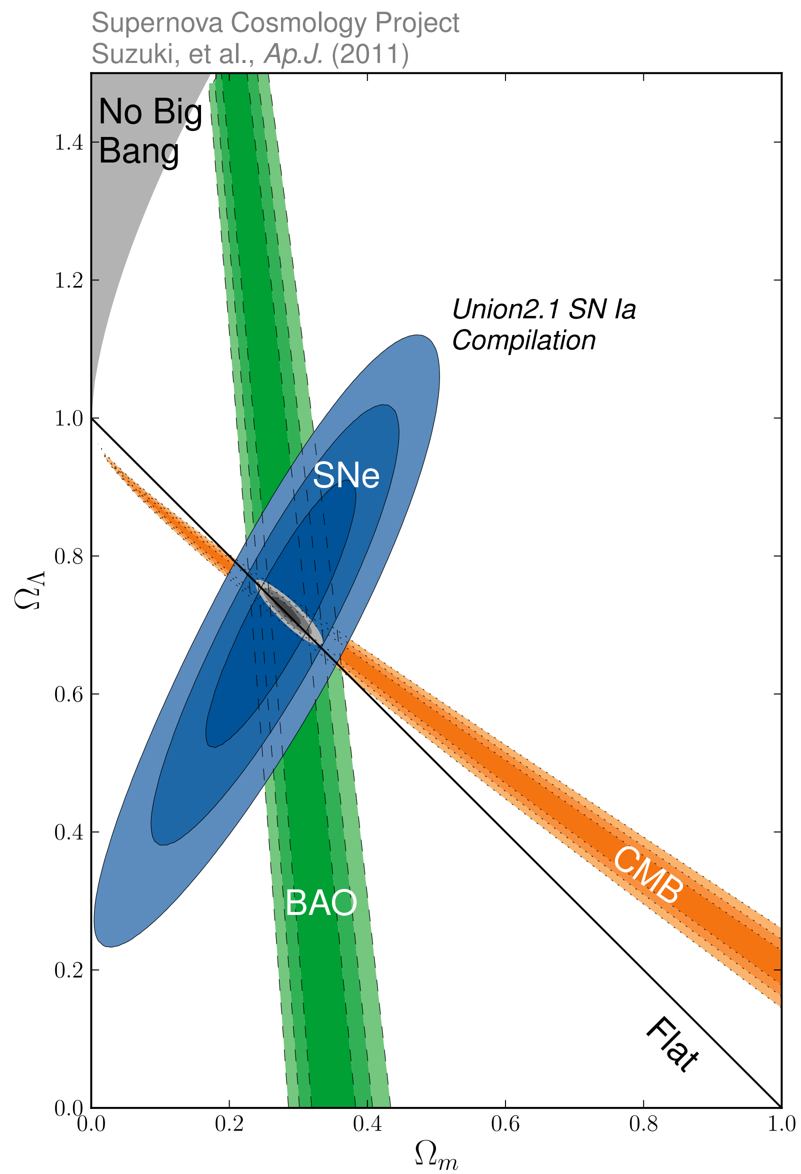
gravity=centrifugal force

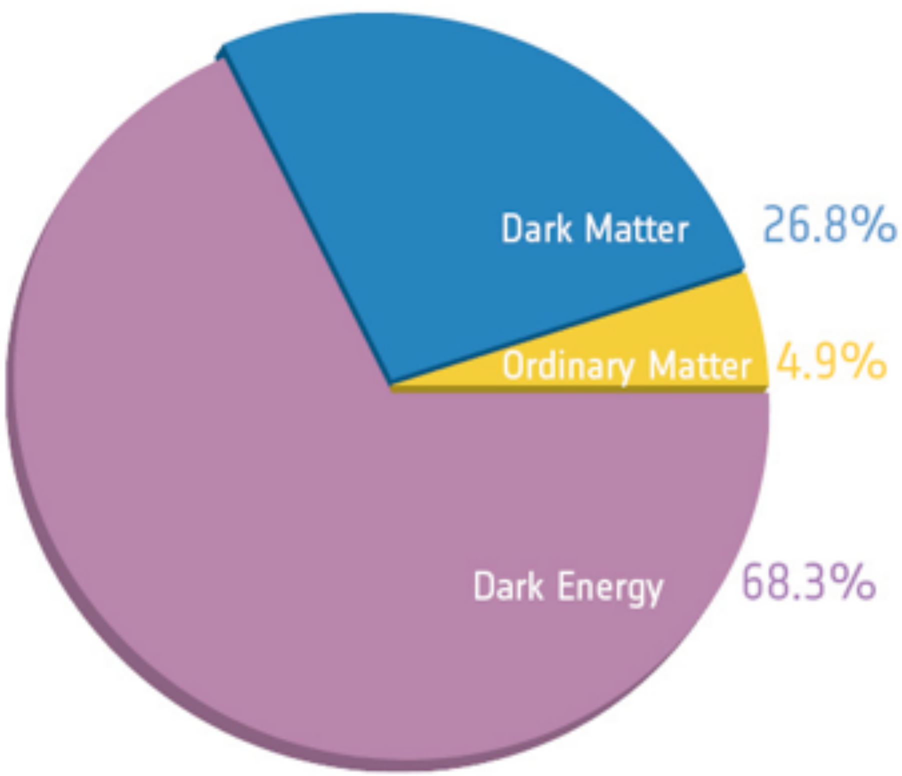
$$v = \sqrt{\frac{GM}{r}}$$

$\Rightarrow$ $M \propto r$ or $\rho(r) \propto \frac{1}{r^2}$

Dark Matter halo

Dark matter abundance





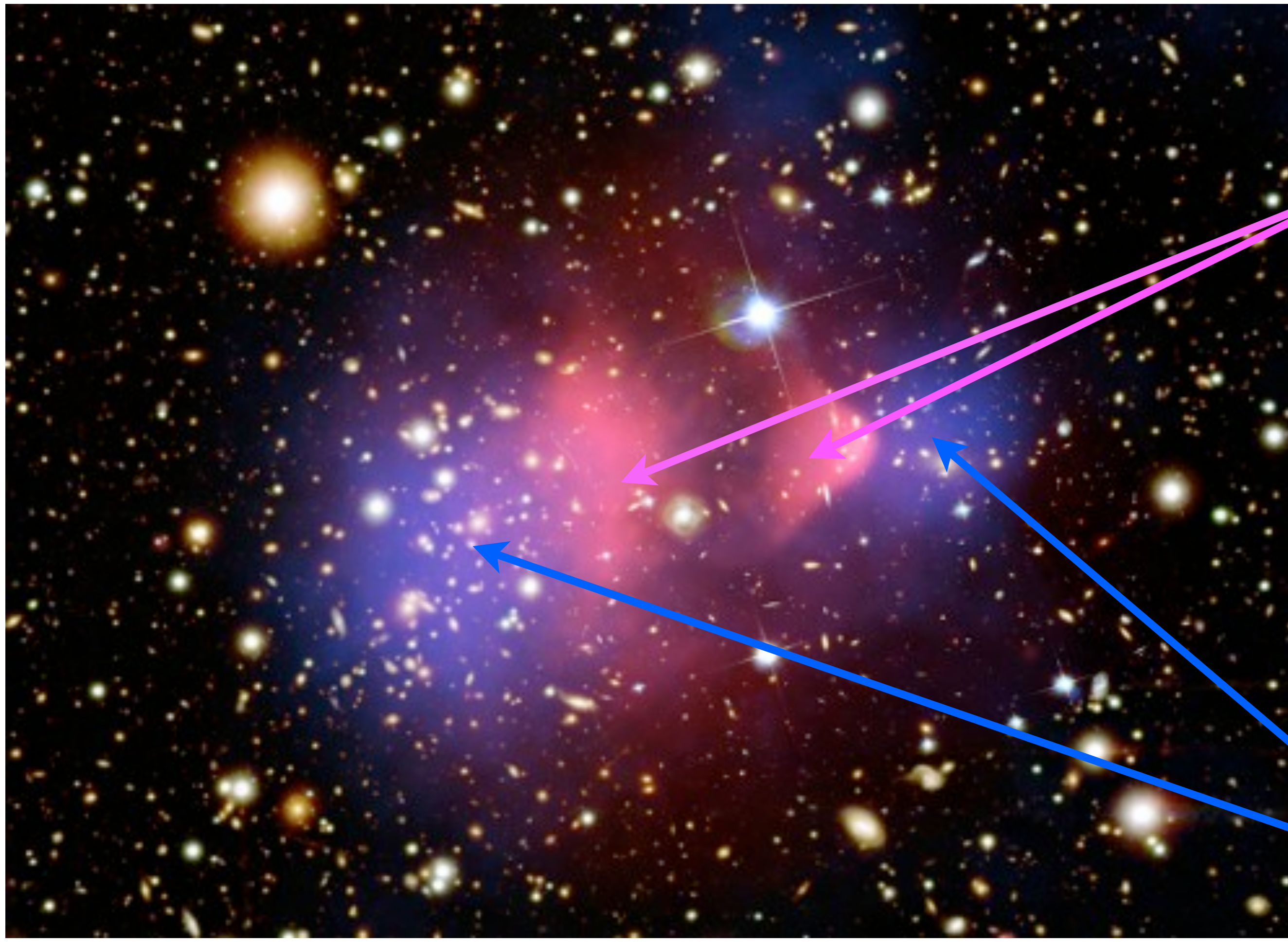
[Planck 2015]

- Cosmic Microwave Background + Supernovae + BAO(SDSS-Large scale structure).

Visible matter ~ just 15% of Universe's matter.

Colliding clusters

[Chandra X-ray (2006)]



Hot gas(X-ray)

(gets slow during collision)

DM(gravitational lensing), stars (visible)

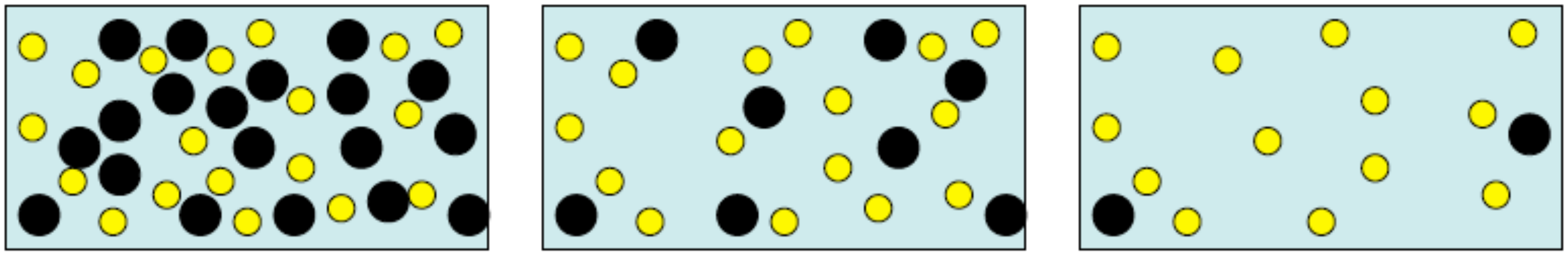
(unaltered during collision)

Bullet cluster

center of total mass ≠ center of baryonic mass

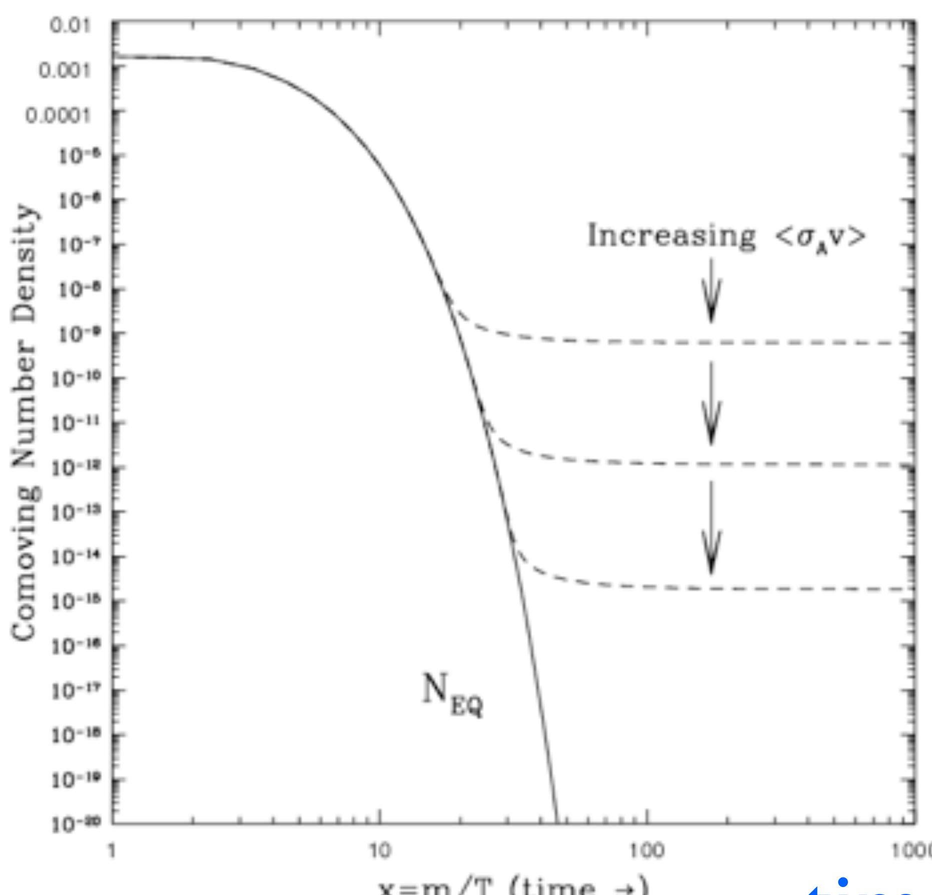
⇒ Bound on self-interaction: $\sigma_{\rm self}/m_\chi \lesssim 1{\rm cm}^2/{\rm g}$

Thermal dark matter



T >> M T ≈ M T << M

DM number



time

[Lee,Weinberg(1977)]

Equilibrium: $\chi\chi \leftrightarrow \mathrm{SM\ SM}$

Decoupling: “Freeze-out”

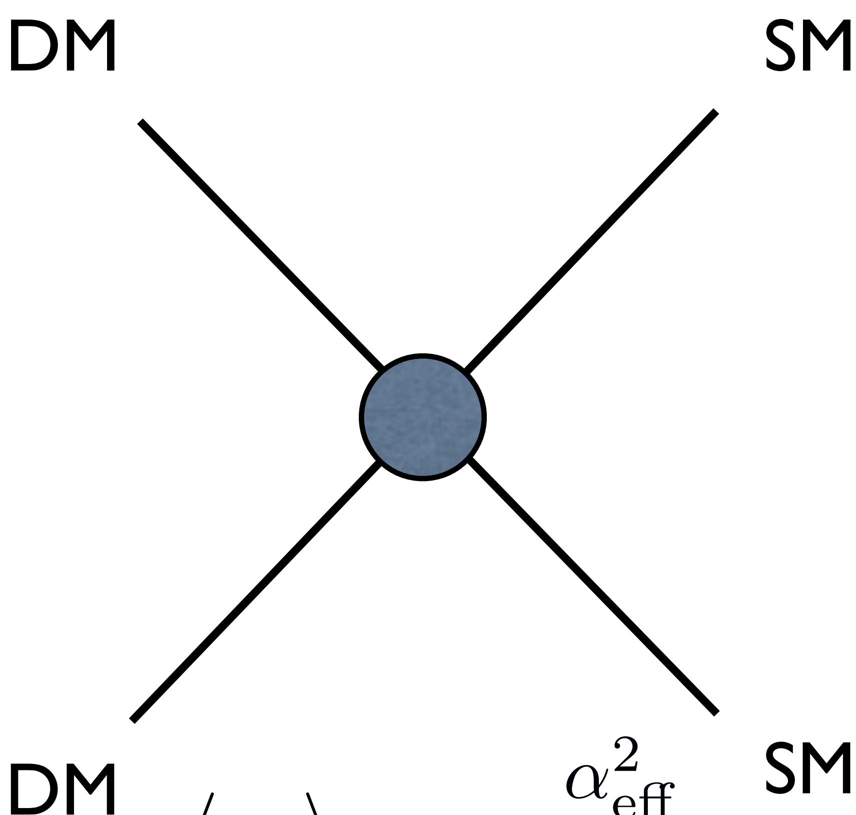
$$t = H^{-1} < t_{\rm int} = (n_{\rm DM}\langle\sigma v\rangle_{\rm ann})^{-1}$$

expansion time interaction time

$$\Rightarrow \quad \Omega_{\rm DM} = 0.3\left(\frac{3\times 10^{-26}\mathrm{cm}^3/\mathrm{s}}{\langle\sigma v_A\rangle}\right)$$

WIMP paradigm

- WIMP DM density depends on 2→2 annihilation processes with weak interactions.



$$\langle \sigma v \rangle_{\rm ann} = \frac{\alpha_{\rm eff}^2}{m_{\rm DM}^2}$$

WIMP freeze-out:

$$\Gamma_{\rm ann} = n_{\rm DM} \langle \sigma v \rangle_{\rm ann} \sim H = 0.33 g_*^{1/2} \frac{T_F^2}{M_P}$$

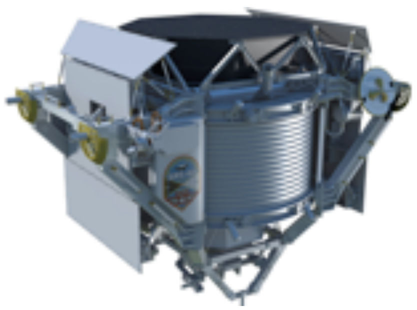
$$\rho_{\rm DM} = 5.4 m_p \eta_p \, s, \;\; T_{\rm eq} = 10 m_p \eta_p;$$

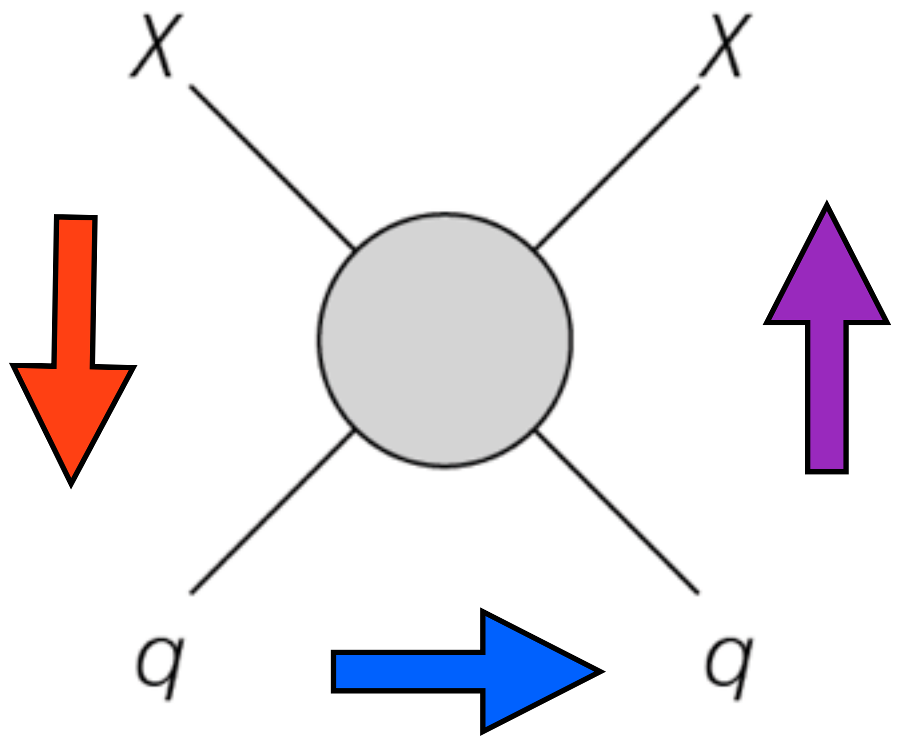
$$n_{\rm DM} = \frac{\kappa T_{\rm eq} T_F^3}{m_{\rm DM}}, \quad \kappa \simeq 0.54 \cdot \frac{2\pi^2}{45} g_{*s} \simeq 10.8.$$

$$\Rightarrow \quad m_{\rm DM} = \alpha_{\rm eff} \left(\frac{5.35\kappa}{x_F g_*^{1/2}} T_{\rm eq} M_P \right)^{1/2}$$

$$\alpha_{\rm eff} \sim 0.1 \quad \longrightarrow \quad m_{\rm DM} \sim 100\,{\rm GeV}.$$

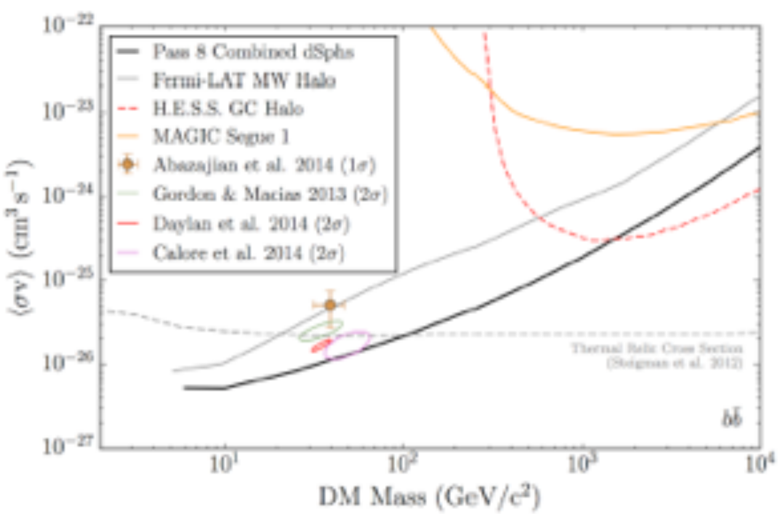
Testing WIMP

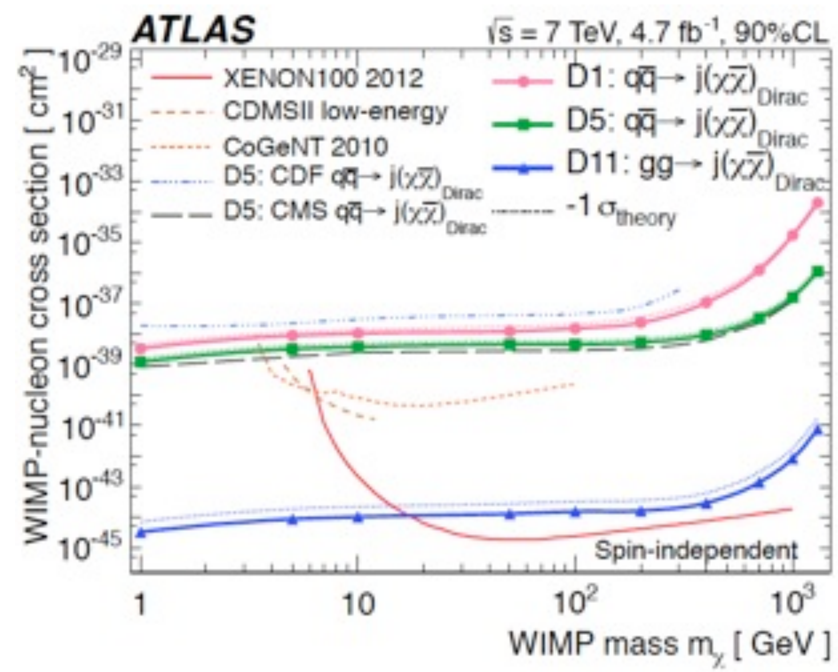




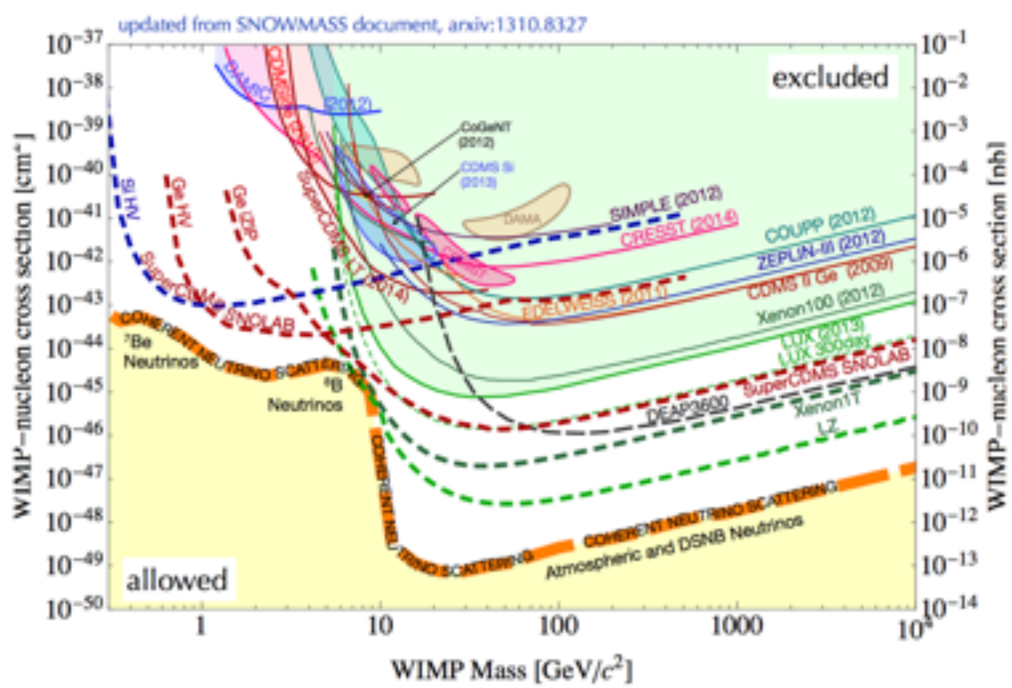
Particle colliders

Indirect detection





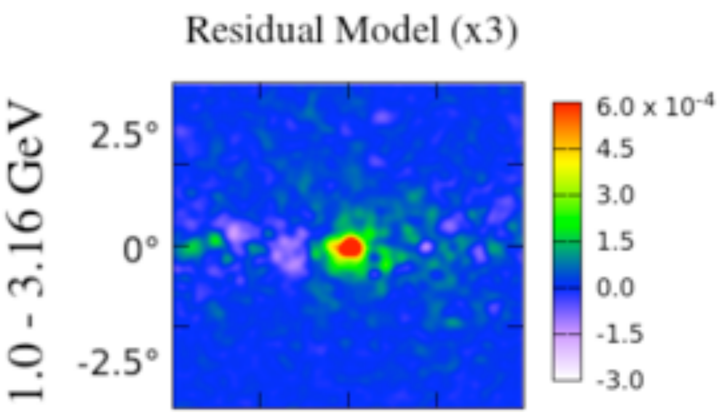
Direct detection

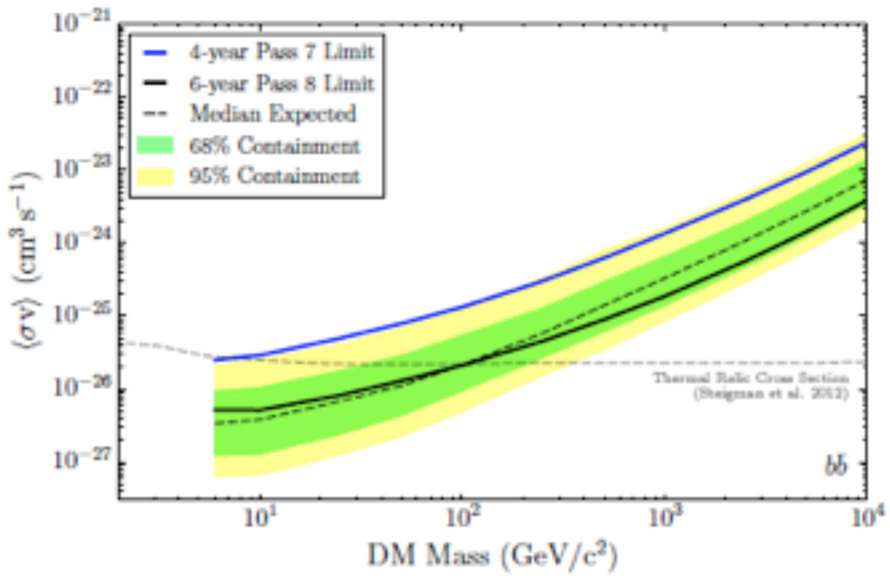


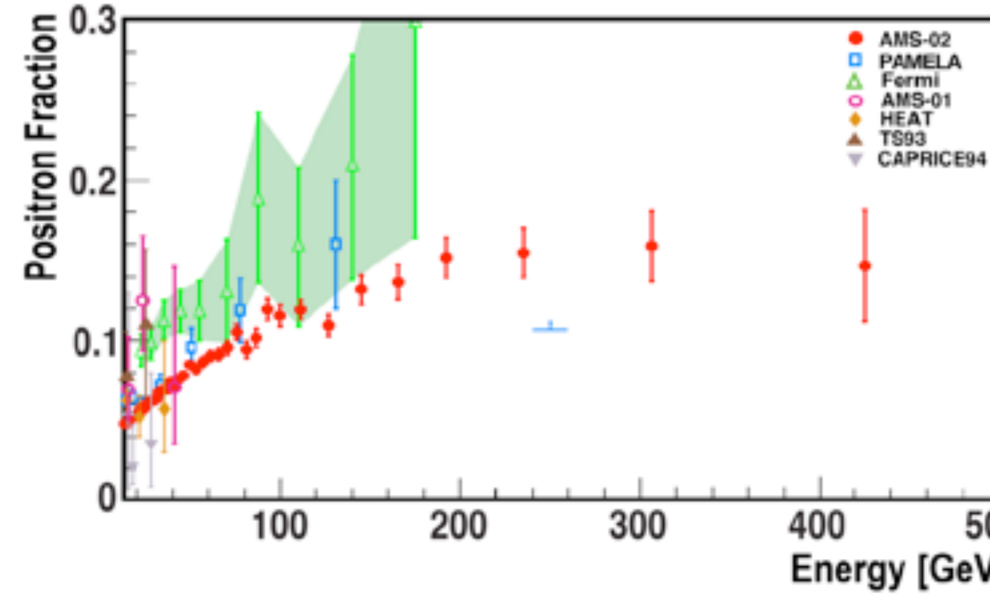
Limits or excesses?

Fermi-LAT: γ-ray
(G.C., dwarf galaxies)

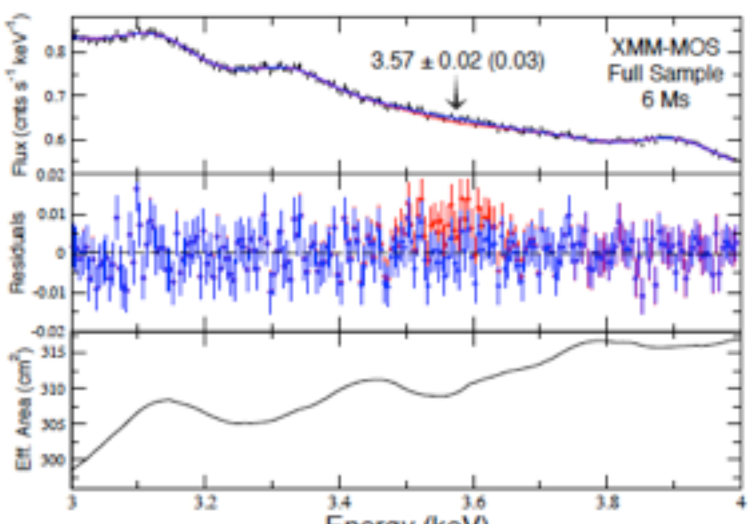
AMS-02: $e^+,\ \bar{p}$

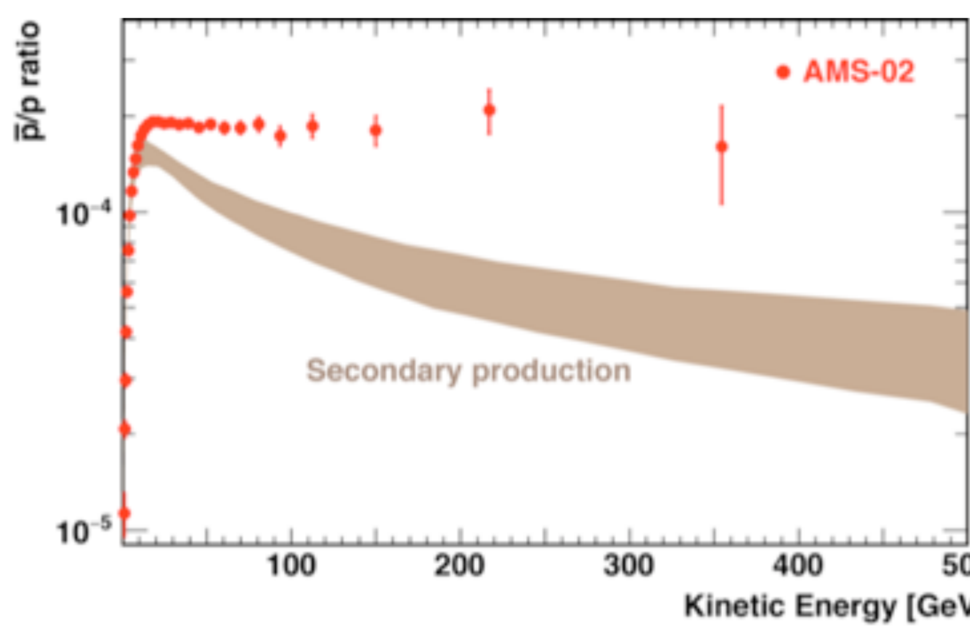






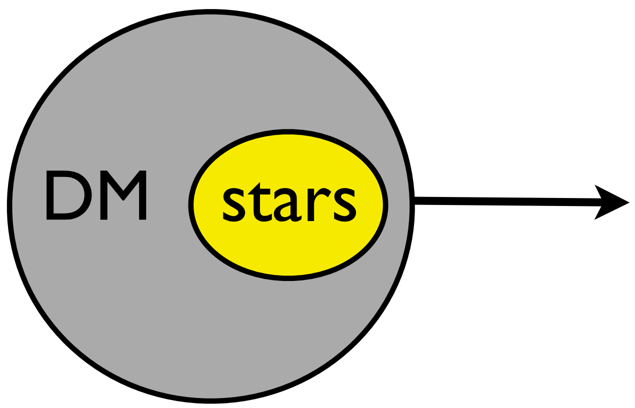
XMM-Newton: X-ray
(galaxy clusters)





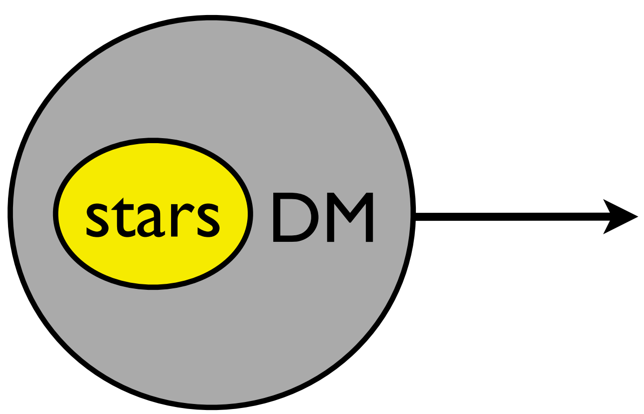
DM self-interactions

- DM self-interactions could induce a separation of DM subhalo from bounded stars.



Long-range forces: "drag force"

⇒ DM subhalo behind stars



Contact interactions: large momentum transfer

⇒ Stars behind DM subhalo

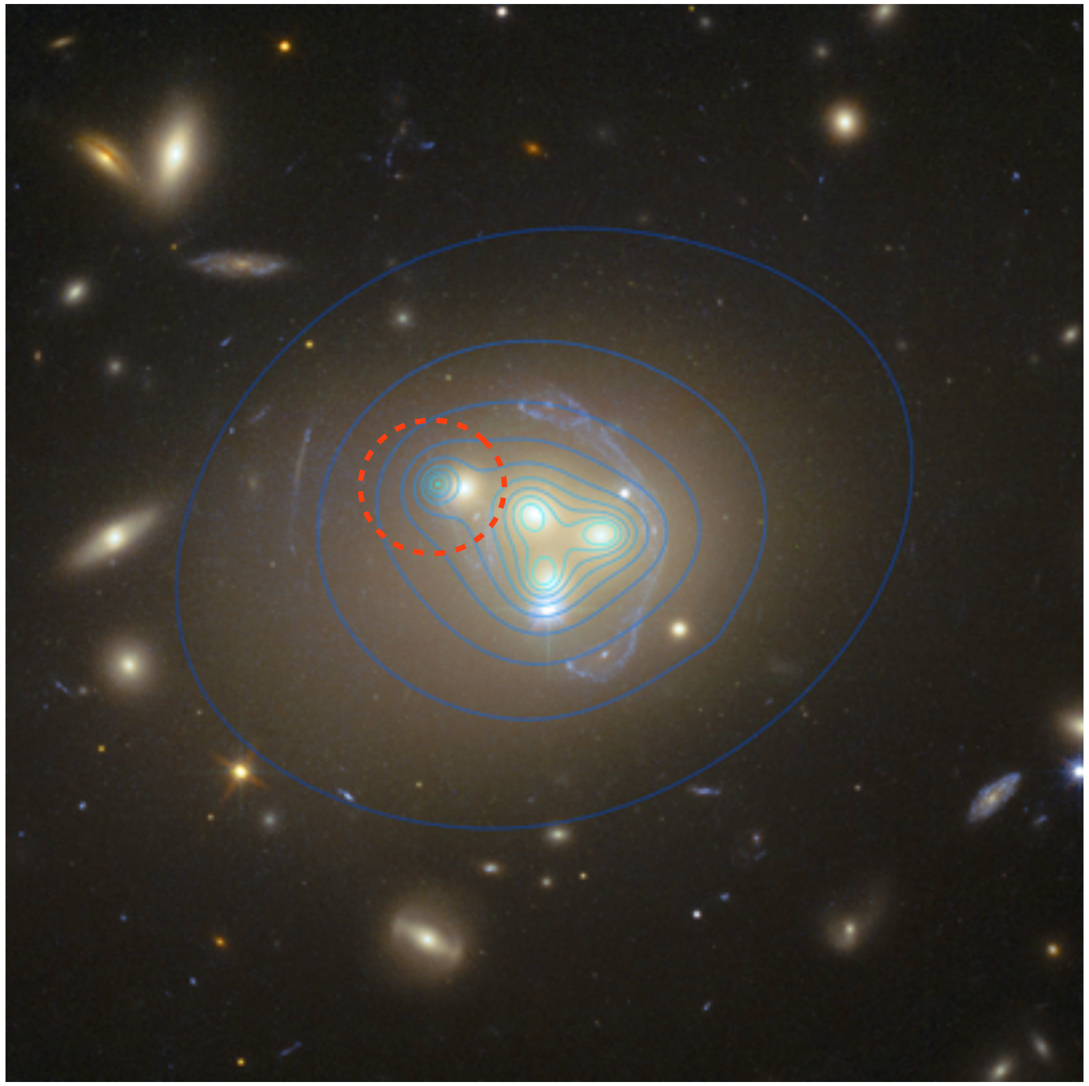
Bullet cluster: no separation of DM subhalo

⇒ $\sigma_{\rm self}/m_{\rm DM} < 1{\rm cm}^2/g$ (about neutron cross section)

WIMP DM: $\sigma_{\rm self}/m_{\rm WIMP} \sim 10^{-11}{\rm GeV}^{-3} \sim 10^{-14}{\rm cm}^2/g.$

Abell 3827

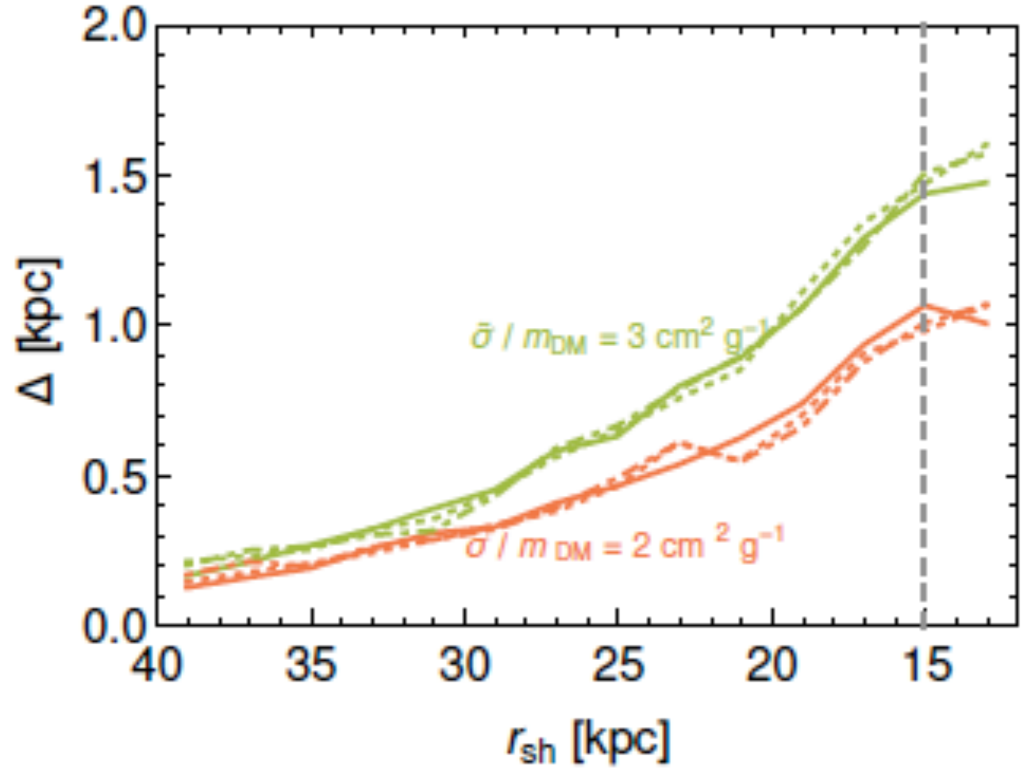
- For four colliding galaxies (not clusters) observed by Hubble Telescope, one of subhalo lags behind the galaxy.



DM subhalo separation:

$$\Delta = 1.62^{+0.47}_{-0.49}\,\mathrm{kpc}$$

[Massey et al(2015)]



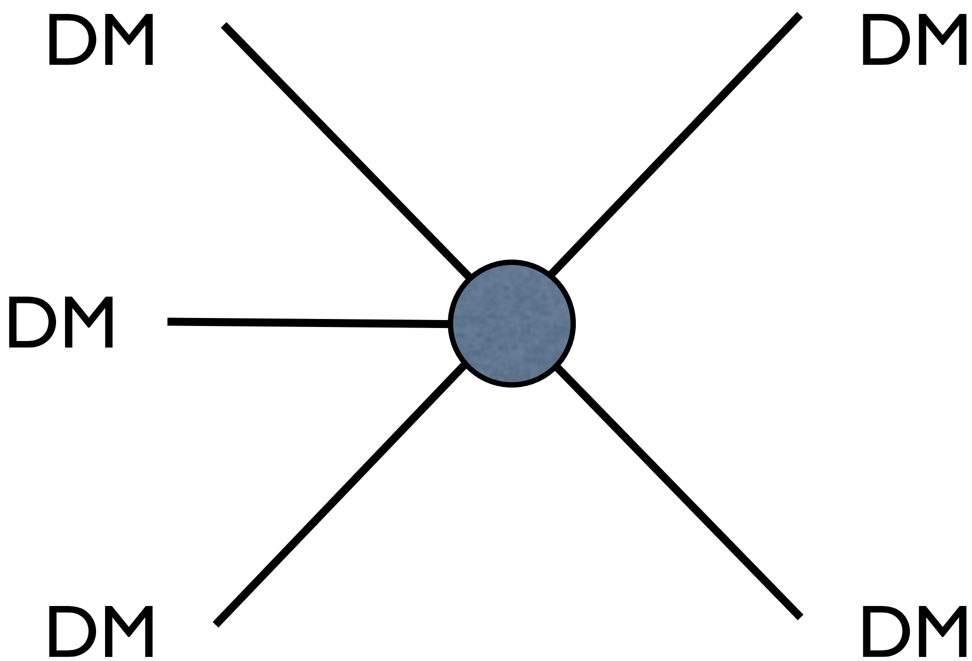
[Kahlhoefer etal (2015)]

Required DM self-interaction in tension with Buller cluster.

SIMP paradigm

- Strong Interacting Massive Particle is a thermal DM, changing its number due to 3→2 self-annihilation.

[Hochberg et al, 2014]



Freeze-out:

$$\Gamma_{3\to 2} = n_{\rm DM}^2 \langle \sigma v^2 \rangle_{3\to 2} \sim H(T_F)$$

$$\langle \sigma v^2 \rangle_{3\to 2} = \frac{\alpha_{\rm eff}^3}{m_{\rm DM}^5}$$

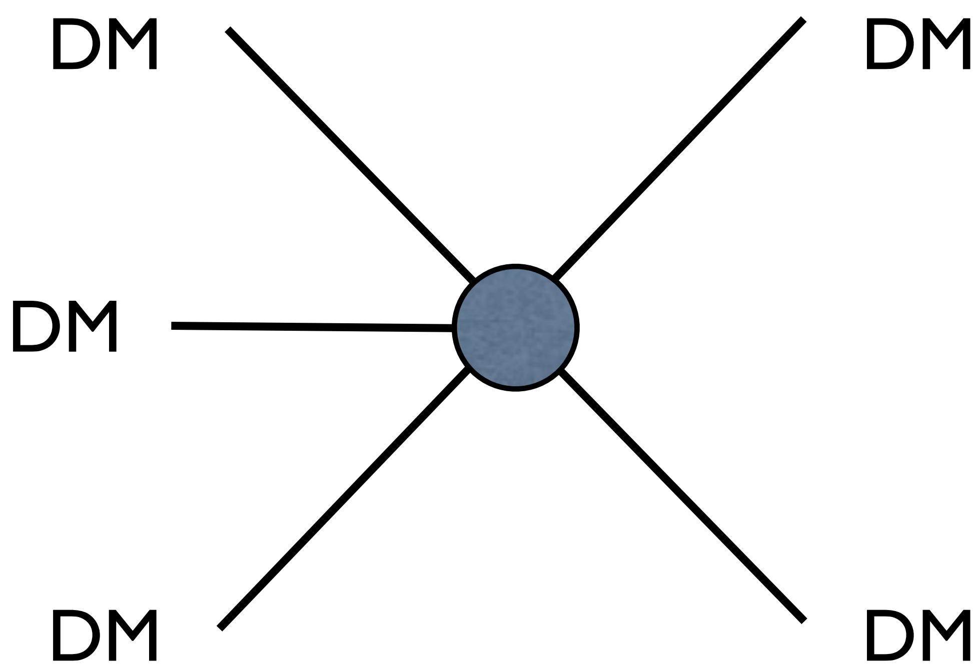
$$\Rightarrow \quad m_{\rm DM} = \alpha_{\rm eff} \left(\frac{5.35\kappa^2}{x_F^4 g_*^{1/2}} T_{\rm eq}^2 M_P \right)^{1/3} \quad \text{or} \quad \Omega_{\rm DM} = 0.3 \left(\frac{3\times 10^{-26}{\rm cm}^3/{\rm s}}{n_{\rm DM}\langle \sigma v^2 \rangle} \right)$$

$(\kappa \simeq 2.55)$

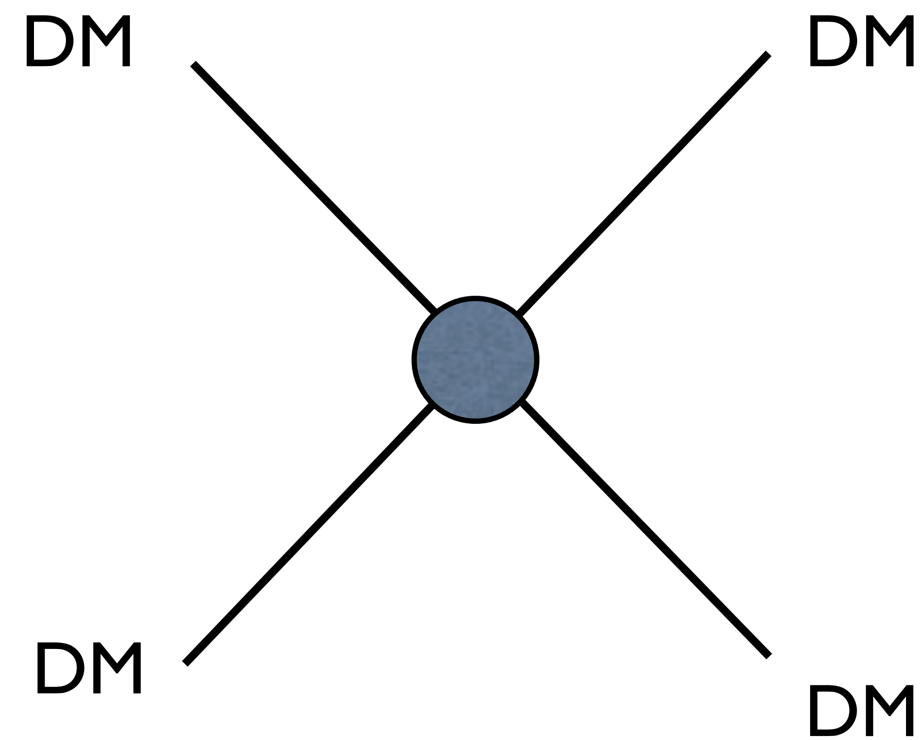
$$\alpha_{\rm eff} = 1 - 30 \quad \Rightarrow \quad m_{\rm DM} \sim 10{\rm MeV} - 1\,{\rm GeV}$$

DM self-interaction

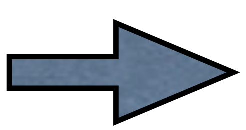
- SIMP DM predicts large DM self-interactions.



$\alpha_{\rm eff} \sim 1$



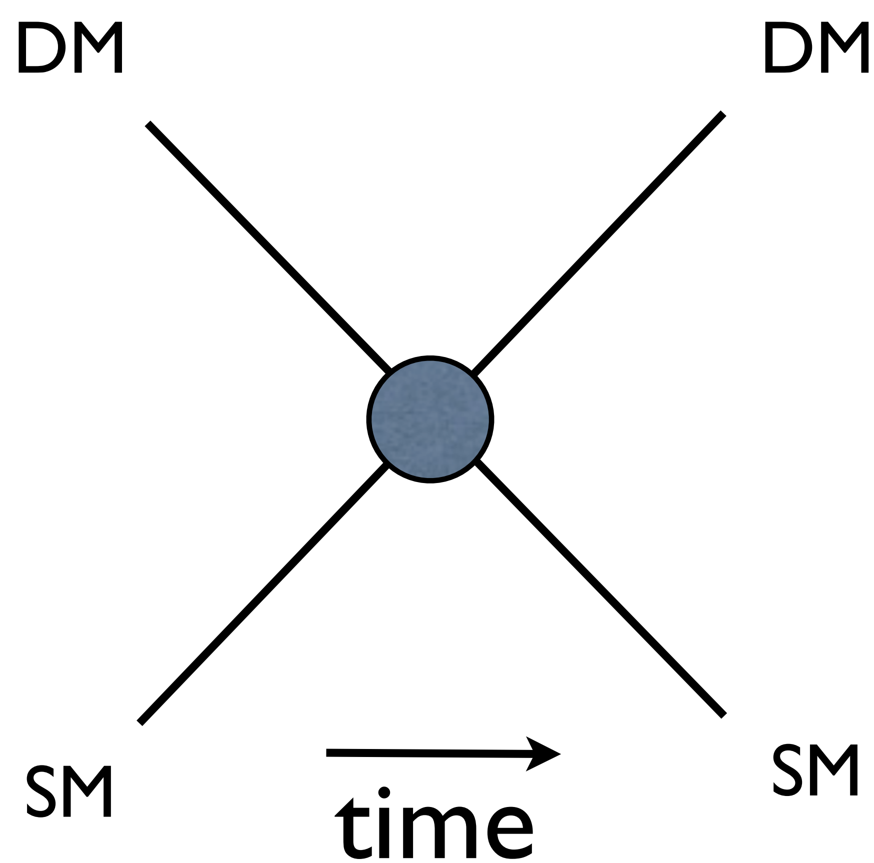
Large DM self-interactions



Constrained by Bullet cluster and spherical halo shapes.

SIMP in kinetic equilibrium

- Kinetic equilibrium of SIMP needs a SM coupling.



$$\langle\sigma v\rangle_{\rm kin} = \frac{\epsilon_1^2}{m_{\rm DM}^2};$$

$$n_{\rm SM}\langle\sigma v\rangle_{\rm kin} > H(T_F)$$

$$\Rightarrow \quad \epsilon_1 \gtrsim 0.9\times 10^{-9}\alpha_{\rm eff}^{1/2}$$

- But, 2→2 DM annihilation is subdominant only if

$$\langle\sigma v\rangle_{\rm ann} = \frac{\epsilon_2^2}{m_{\rm DM}^2};$$

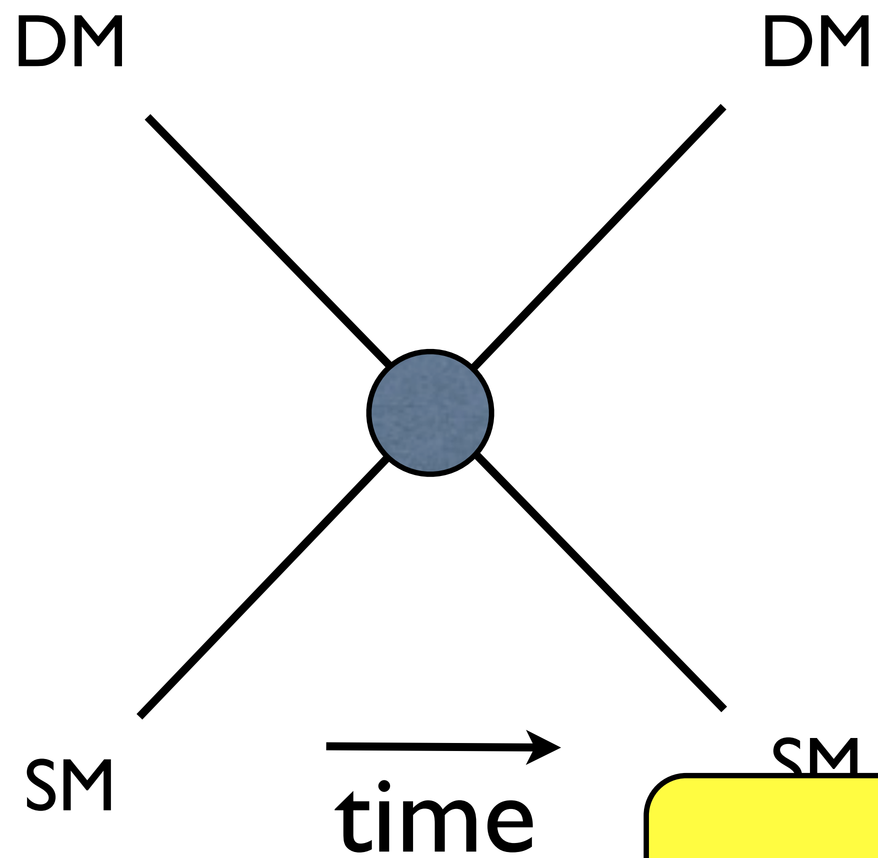
$$n_{\rm SM}\langle\sigma v\rangle_{\rm kin} < n_{\rm DM}\langle\sigma v^2\rangle_{3\to 2}$$

$$\Rightarrow \quad \epsilon_1 \lesssim 2.4\times 10^{-6}\alpha_{\rm eff}$$

$$\epsilon_2 \sim \epsilon_1$$

SIMP in kinetic equilibrium

- Kinetic equilibrium of SIMP needs a SM coupling.



$$\langle\sigma v\rangle_{\rm kin} = \frac{\epsilon_1^2}{m_{\rm DM}^2};$$

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$$\Rightarrow \quad \epsilon_1 \gtrsim 0.9\times 10^{-9}\alpha_{\rm eff}^{1/2}$$

SIMP conditions

- But, 2→2 DM … ant only if

$$\langle\sigma v\rangle_{\rm ann} = \frac{\epsilon_2^2}{m_{\rm DM}^2};$$

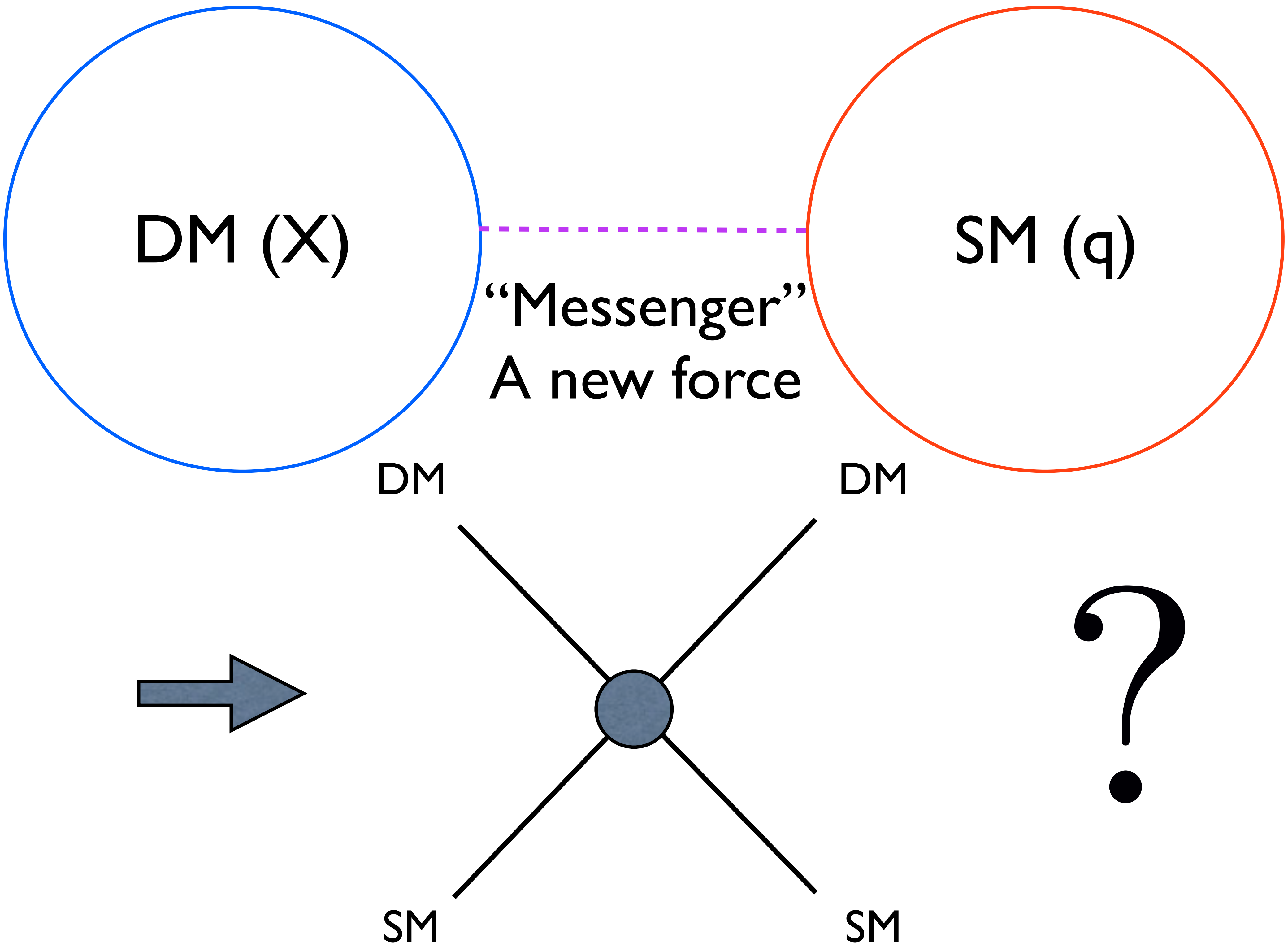
$$n_{\rm SM}\langle\sigma v\rangle_{\rm kin} < n_{\rm DM}\langle\sigma v^2\rangle_{3\to 2}$$

$$\Rightarrow \quad \epsilon_1 \lesssim 2.4\times 10^{-6}\alpha_{\rm eff}$$

$$\epsilon_2 \sim \epsilon_1$$

DM messengers

- Messenger particles mediate between DM and the SM.



SIMP DM from hidden QCD

Dark mesons & WZW term

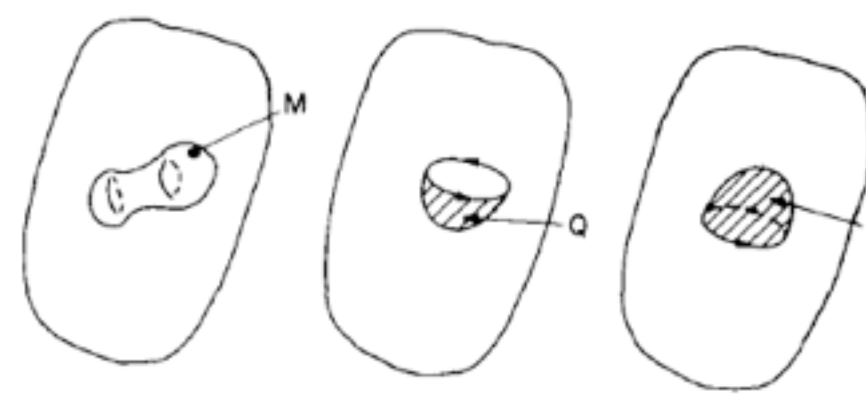
- Dark flavor symmetry G=SU(N_f)x SU(N_f) is SSB into diagonal H=SU(N_f) by SU(N_c) QCD-like condensation.
- Effective action for Goldstone bosons contains a 5-point self-interaction from Wess-Zumino-Witten term for $\pi_5(G/H)=Z$ (i.e. $N_f \geq 3$).

[Wess, Zumino, 1971;Witten, 1983]

$$U = e^{2i\pi/F}, \quad \pi \equiv \pi^a T^a$$

$$\mathcal{L}_{WZW} = \frac{2N_c}{15\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[\pi\partial_\mu\pi\partial_\nu\pi\partial_\rho\pi\partial_\sigma\pi]$$

$$N_f = 3:\ \pi = \frac{\sqrt{2}}{F}\begin{bmatrix} \frac{1}{\sqrt{2}}\tilde{\pi}_0 + \frac{1}{\sqrt{6}}\tilde{\eta}^0 & \tilde{\pi}^+ & \tilde{K}^+ \\ \tilde{\pi}^- & -\frac{1}{\sqrt{2}}\tilde{\pi}_0 + \frac{1}{\sqrt{6}}\tilde{\eta}^0 & \tilde{K}^0 \\ \tilde{K}^- & \overline{\tilde{K}}^0 & -\sqrt{\frac{2}{3}}\tilde{\eta}^0 \end{bmatrix}.$$

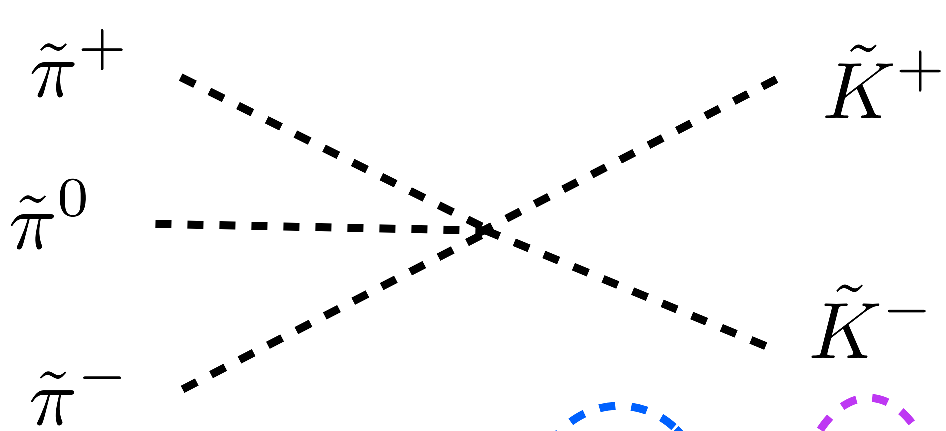


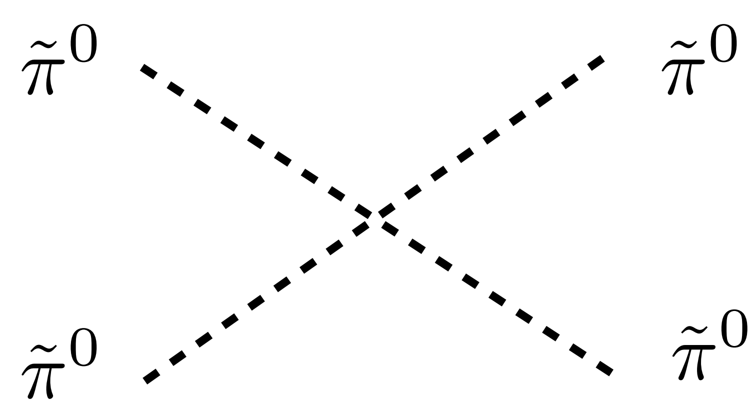
N_C : topological invariant of 5-sphere (Q+Q') in SU(3)

Flavor symmetry ensures stability of dark mesons, natural candidates for SIMP.

SIMP dark mesons

- Large color group leads to strong 5-point interactions while satisfying bounds on self-interactions (e.g. Bullet cluster, halo shape.) [Hochberg et al, 2014]



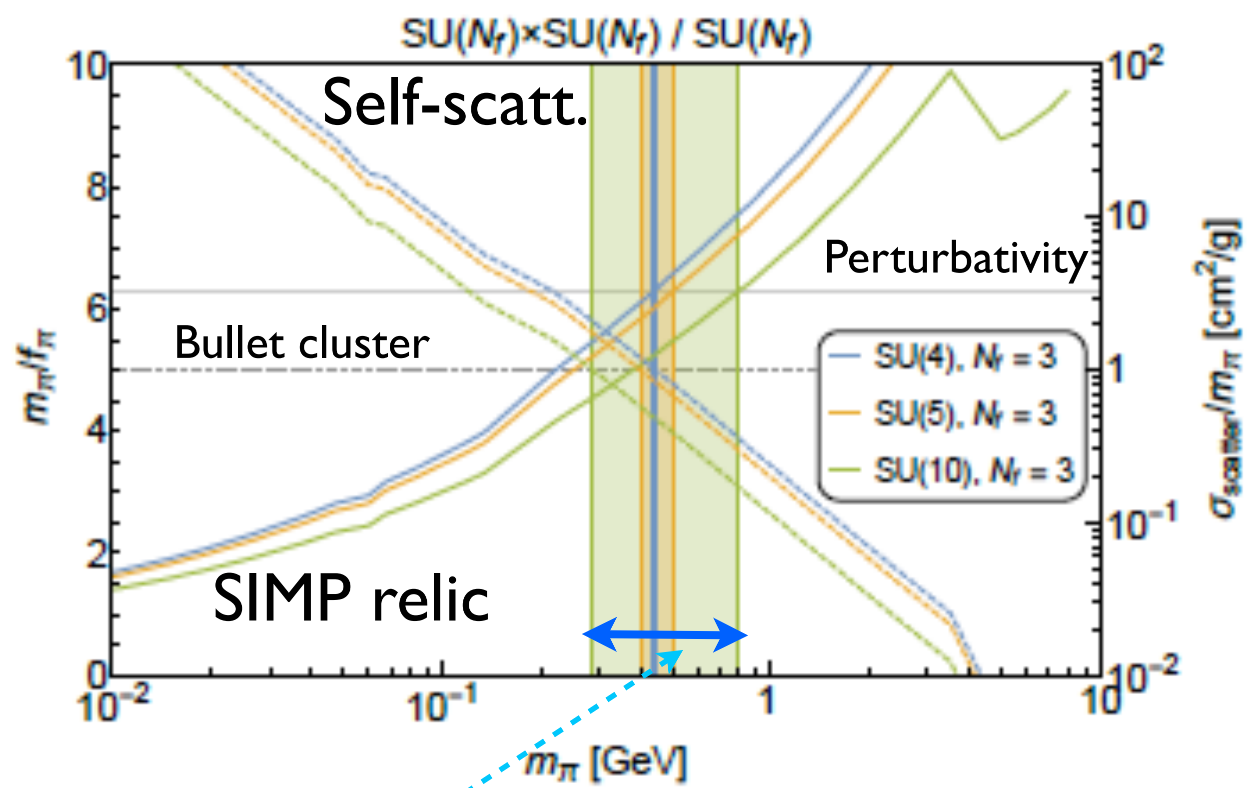


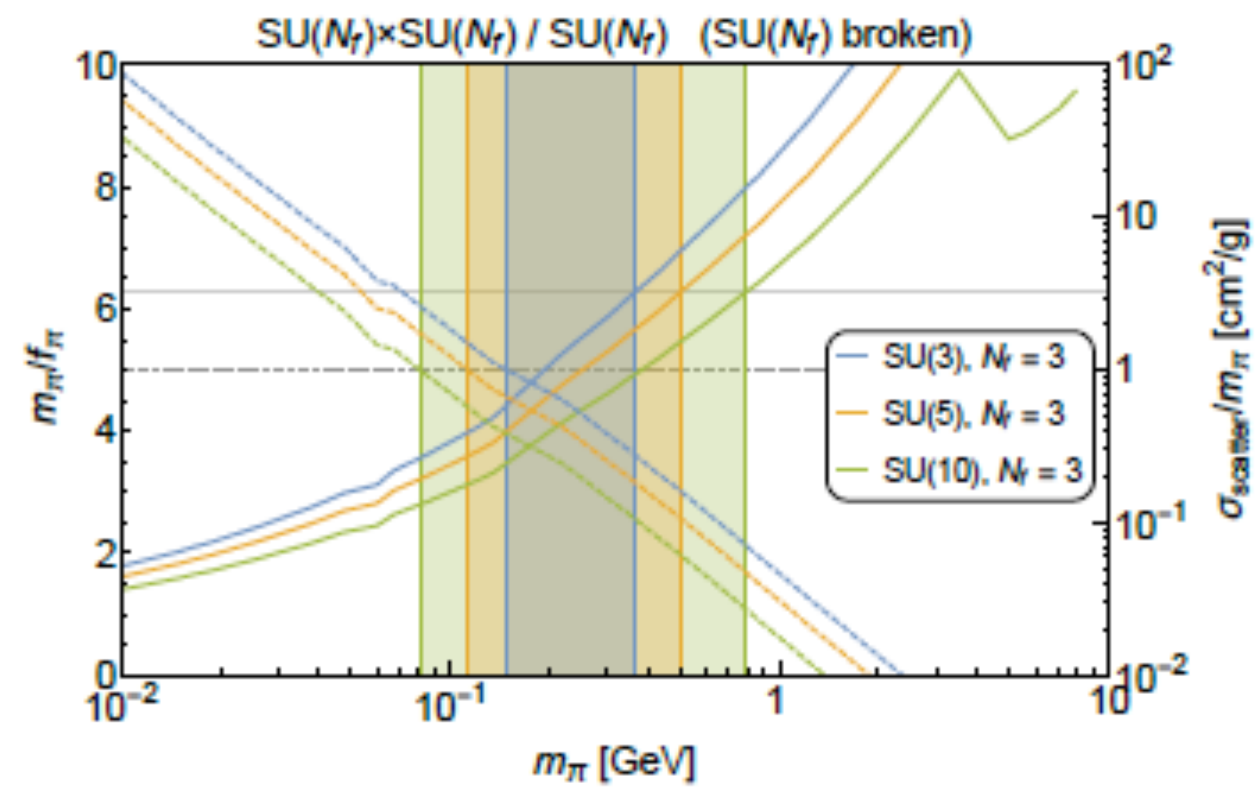
$$\langle \sigma v^2 \rangle_{3\to 2} = \frac{5\sqrt{5} N_c^2 m_\pi^5}{2\pi^5 F^{10}} \frac{t^2}{N_\pi^3} \left(\frac{T_F}{m_\pi} \right)^2, \qquad \sigma_{\rm self} = \frac{m_\pi^2}{32\pi F^4} \frac{a^2}{N_\pi^2}$$

~const ~const

G_c	G_f/H	N_π	t^2	$N_f^2 a^2$
$SU(N_c)$	$\frac{SU(N_f)\times SU(N_f)}{SU(N_f)}$ $(N_f \geq 3)$	$N_f^2 - 1$	$\frac{4}{3} N_f (N_f^2-1)(N_f^2-4)$	$8(N_f-1)(N_f+1)(3N_f^4 - 2N_f^2 + 6)$
$SO(N_c)$	$SU(N_f)/SO(N_f)$ $(N_f \geq 3)$	$\frac{1}{2}(N_f+2)(N_f-1)$	$\frac{1}{12} N_f (N_f^2-1)(N_f^2-4)$	$(N_f-1)(N_f+2)(3N_f^4 + 7N_f^3 - 2N_f^2 - 12N_f + 24)$
$Sp(N_c)$	$SU(2N_f)/Sp(2N_f)$ $(N_f \geq 2)$	$(2N_f+1)(N_f-1)$	$\frac{2}{3} N_f (N_f^2-1)(4N_f^2-1)$	$4(N_f-1)(2N_f+1)(6N_f^4 - 7N_f^3 - N_f^2 + 3N_f + 3)$

SIMP parameter space





consistent DM masses

[Hochberg, Kuflik, Murayama, Volansky, Wacker, 1411.3727]

Bullet cluster, Halo shape $\sigma_{\rm self}/m_{\rm DM} < 1\,{\rm cm}^2/{\rm g}$

Perturbativity $m_\pi/f_\pi < 2\pi$

$N_c>3$ is required due to bounds on self-scattering.

Similar results for $SU(N_f)/SO(N_f)$ or $SU(2N_f)/Sp(2N_f)$.

Dark Z' and WZW

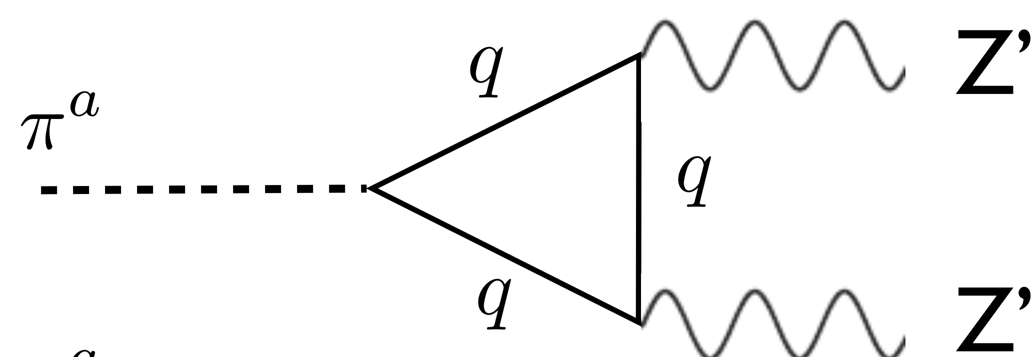
[HML, M.Seo, 2015]

- Dark quarks are vector-like under local dark U(1).
- WZW is modified with the local U(1).

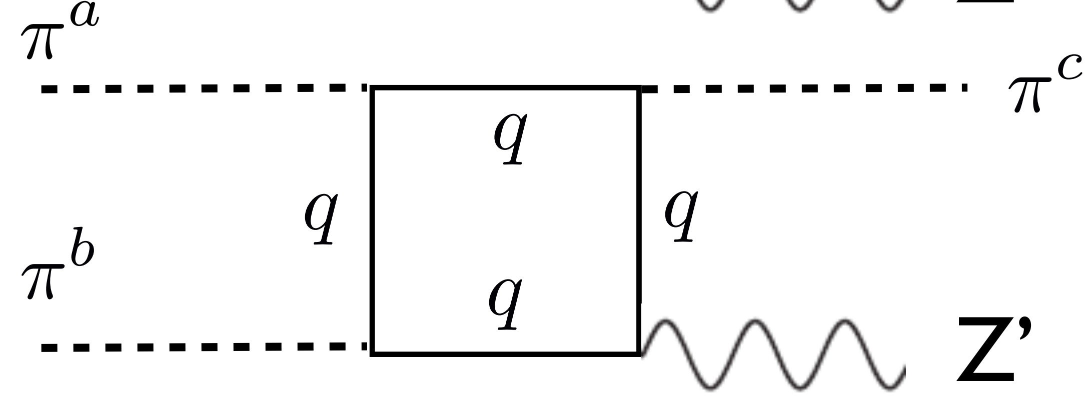
$$
\begin{aligned}
S =& S_0(D_\mu U, D_\mu U^{-1}) + S_{\rm WZW}(U,U^{-1}) - eN_c \int d^4x A'_\mu J^\mu \\
&+ \frac{ie^2N_c}{24\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}\partial_\mu A'^\nu A'_\rho {\rm Tr}[Q^2\partial_\sigma U U^{-1} \\
&\qquad + Q^2U^{-1}\partial_\sigma U + QUQU^{-1}\partial_\sigma U U^{-1}],
\end{aligned}
$$

$$
\begin{aligned}
J^\mu =& \frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}{\rm Tr}[Q\partial_\nu U U^{-1}\partial_\rho U U^{-1}\partial_\sigma U U^{-1} \\
&+ QU^{-1}\partial_\nu U U^{-1}\partial_\rho U U^{-1}\partial_\sigma U].
\end{aligned}
$$

- AVV anomalies.

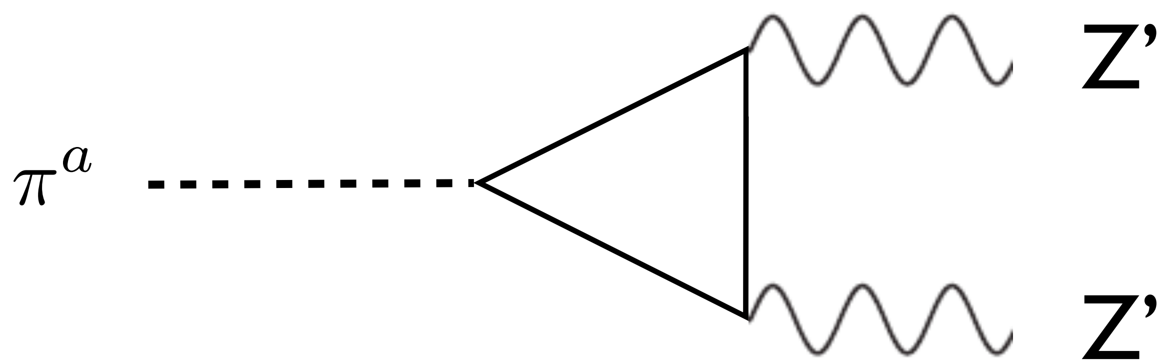


- AAAV anomalies.



Dark charges

- Stability of dark neutral mesons requires the cancellation of AVV anomalies.



$$\propto \mathrm{Tr}(Q_D^2 T^a) = 0$$

if $Q_D^2 = I.$

$\Rightarrow$ $$Q_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ for nonzero meson charges.

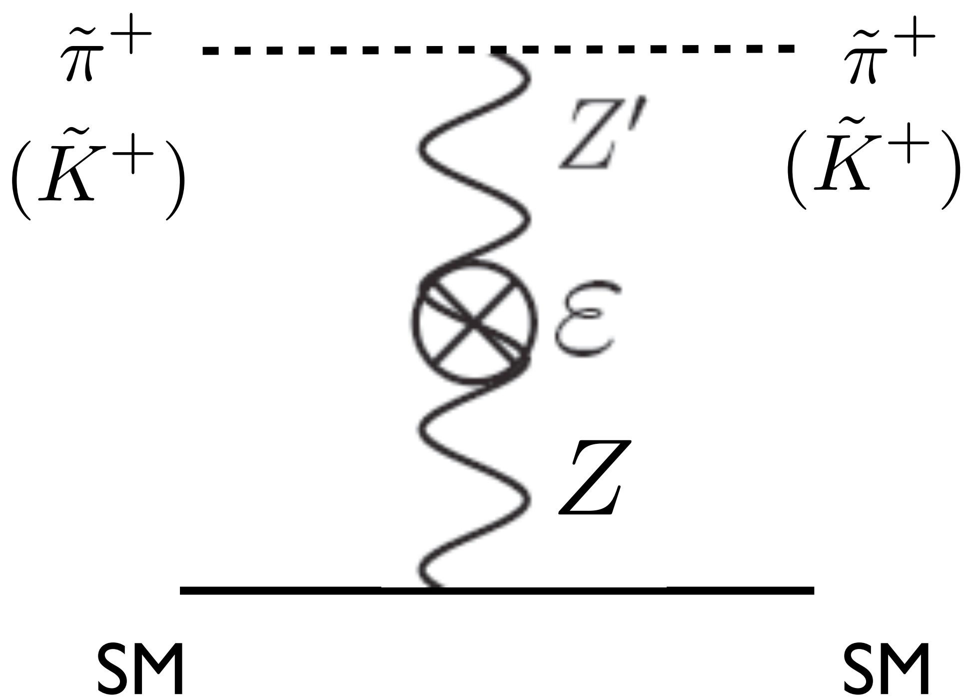
cf. QCD:
Q=diag(2/3,-1/3,-1/3)

$D_\mu U = \partial_\mu U + i g_D [Q_D, U] Z'_\mu; \quad \tilde{\pi}^\pm, \tilde{K}^\pm :$ ±2 charges.

But, AAAV anomalies do not cancel,
leading to π π → πZ'.

Z'-portal for dark mesons

- Dark meson can be in kinetic equilibrium with the SM particles via gauge kinetic mixing.



$$\mathcal{L}_{\rm mix} = -\frac{\varepsilon}{2\cos\theta_W} F'_{\mu\nu}F^{\mu\nu}$$

$$\Rightarrow \quad \langle \sigma v\rangle_{\rm kin} \approx \frac{768\alpha\alpha_D\varepsilon^2}{\pi N_\pi}\frac{m_\pi^2}{m_{Z'}^4}\left(\frac{T_F}{m_\pi}\right).$$

cf. Higgs-portal does not work,
due to small lepton Yukawa couplings.

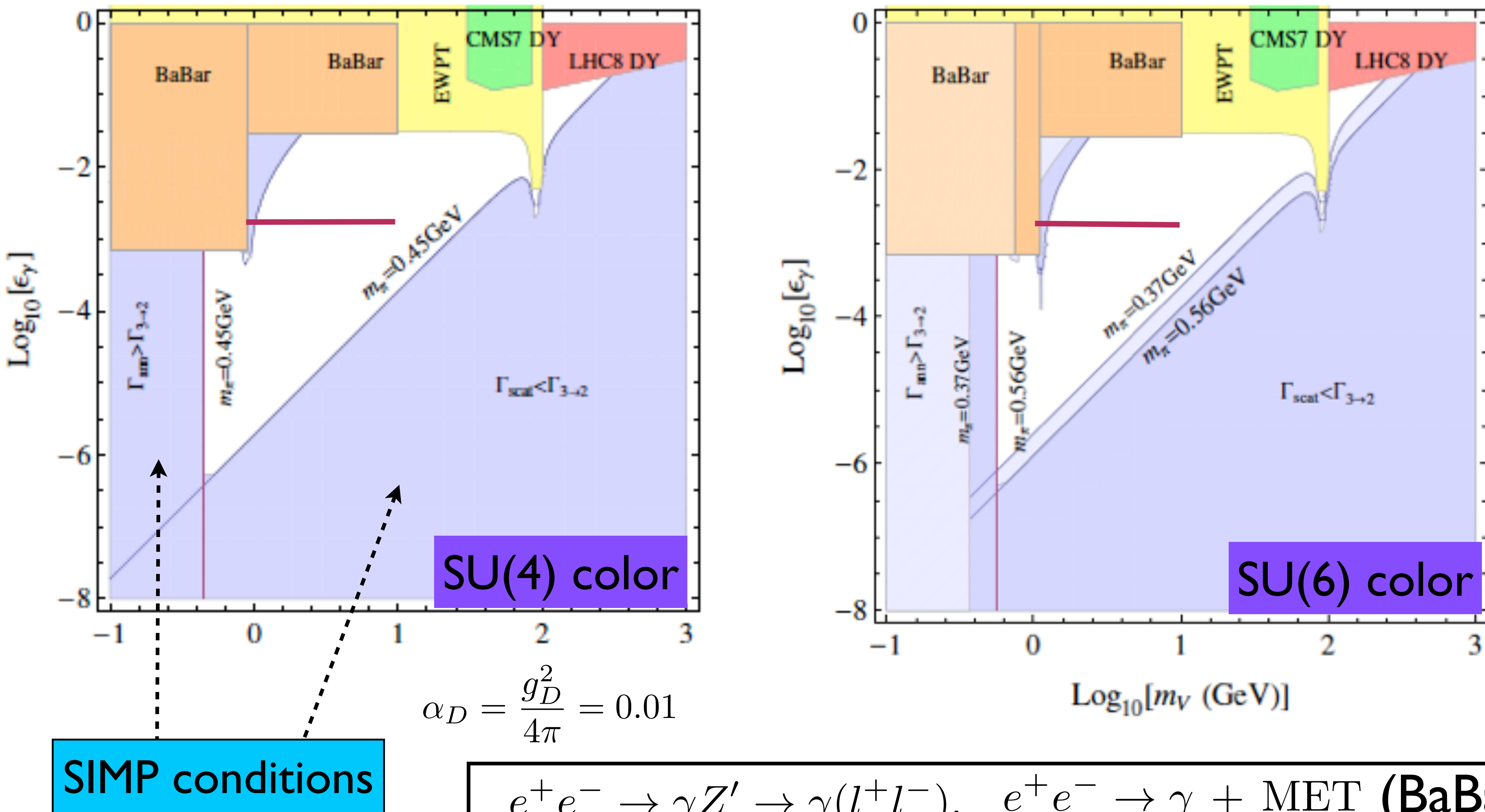
- π π → Z'Z' (πZ') are forbidden for $m_{Z'} > m_\pi$.

- Mass splitting between dark mesons can be less than 10%.

$$\delta\mathcal{L}_{m_\pi} = \alpha_D\Lambda^4{\rm Tr}[Q_D U Q_D U^{-1}] \quad \Rightarrow \quad \delta m_\pi^2 \sim \alpha_D\Lambda^4/F^2 \lesssim 0.01 m_\pi^2,$$

for $\alpha_D \lesssim 0.1$

Bounds on Z’



$$\alpha_D = \frac{g_D^2}{4\pi} = 0.01$$

$e^+e^- \to \gamma Z' \to \gamma(l^+l^-)$, $e^+e^- \to \gamma + \mathrm{MET}$ (BaBar),

$h \to ZZ'$ (CMS 8TeV), Drell-Yan, dileptons.

- SIMP conditions are complementary in constraining Z’ parameters to direct Z’ searches.

SIMP DM from Z_3 symmetry

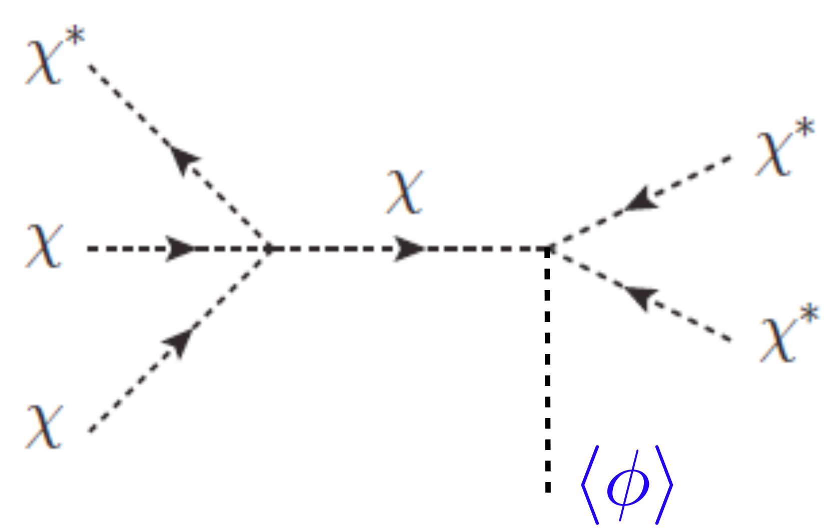
Gauged Z_3 and SIMP

- 5-point SIMP interaction is inconsistent with Z_2.
- Z_3 is the minimal symmetry for stabilizing SIMP, a remnant of a local U(1).
- Built-in Z' gauge boson communicates with the SM via the kinetic mixing.

	ϕ	χ
$U(1)_V$	$+3$	$+1$

Table 1: $U(1)_V$ charges.

$$\langle\phi\rangle = \frac{1}{\sqrt{2}} v' \quad \Rightarrow \quad U(1)_V \to Z_3.$$



A Z_3 Model

- χ: Dark Matter, ϕ: Dark Higgs, V_μ: Dark photon.

$$\mathcal{L} = -\frac{1}{4}V_{\mu\nu}V^{\mu\nu} - \frac{1}{2}\sin\xi\, V_{\mu\nu}B^{\mu\nu} + |D_\mu\phi|^2 + \boxed{|D_\mu\chi|^2} + |D_\mu H|^2 - V(\phi,\chi,H)$$

$$D_\mu\chi = (\partial_\mu - iq_\chi g_D V_\mu)\chi$$

$$V(\phi,\chi,H) = V_{\rm DM} + V_{\rm SM} \text{ with}$$

$$\begin{aligned} V_{\rm DM} &= -m_\phi^2|\phi|^2 + m_\chi^2|\chi|^2 + \lambda_\phi|\phi|^4 + \boxed{\lambda_\chi|\chi|^4} + \lambda_{\phi\chi}|\phi|^2|\chi|^2 \\ &\quad + \boxed{\left(\frac{\sqrt{2}}{3!}\kappa\phi^\dagger\chi^3 + \text{h.c.}\right)} + \lambda_{\phi H}|\phi|^2|H|^2 + \lambda_{\chi H}|\chi|^2|H|^2, \\ V_{\rm SM} &= -m_H^2|H|^2 + \lambda_H|H|^4. \end{aligned}$$

Scalar SIMP DM

- 2→2 annihilation channels are forbidden for heavy dark Higgs and Z'.

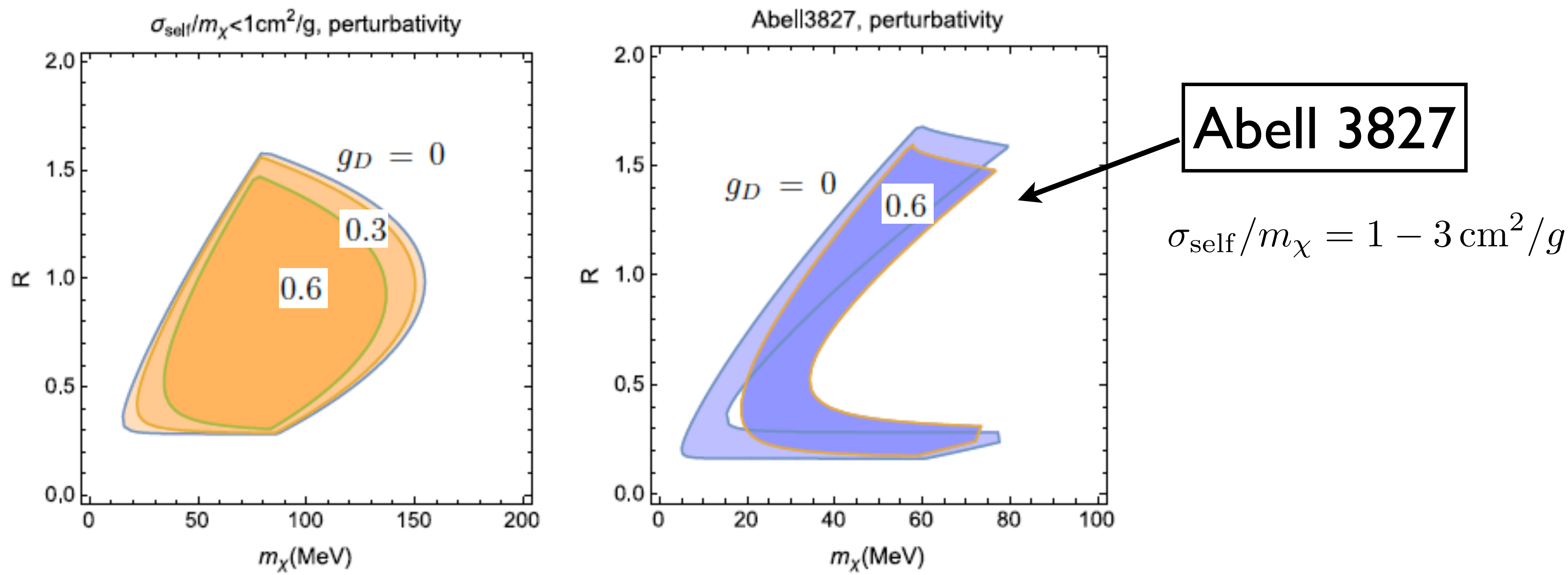
- Boltzmann equation with 3→2 annihilation.

$$\begin{aligned}\frac{dn_{\rm DM}}{dt}+3Hn_{\rm DM} &= -\frac{1}{4}\left(\langle\sigma v_{\rm rel}^2\rangle_{\chi\chi\chi^*\to\chi^*\chi^*}+\langle\sigma v_{\rm rel}^2\rangle_{\chi\chi\chi\to\chi\chi^*}\right)\left(n_{\rm DM}^3-n_{\rm DM}^2 n_{\rm DM}^{\rm eq}\right)\\ &\quad -\frac{1}{2}\langle\sigma v_{\rm rel}\rangle_{\chi\chi^*\to\bar f f}(n_{\rm DM}^2-(n_{\rm DM}^{\rm eq})^2).\end{aligned}$$

$$\begin{aligned}\langle\sigma v_{\rm rel}^2\rangle_{3\to2} &= \frac{1}{4}\left(\langle\sigma v_{\rm rel}^2\rangle_{\chi\chi\chi^*\to\chi^*\chi^*}+\langle\sigma v_{\rm rel}^2\rangle_{\chi\chi\chi\to\chi\chi^*}\right)\\ &= \frac{\sqrt{5}R^2}{1536\pi m_\chi^5}\Big\{3\Big(2\lambda_\chi+9R^2+25g_D^2(1+r)^{-1}\Big)^2\\ &\quad +\frac{1}{16}\Big(74\lambda_\chi-117R^2-200g_D^2(1+r)^{-1}\Big)^2\Big\}\equiv\frac{\alpha_{\rm eff}^3}{m_\chi^5}.\end{aligned}$$

$$R\equiv\sqrt{2}\kappa v'/(6m_\chi)\ \text{and}\ r\equiv m_{Z'}^2/m_\chi^2.$$

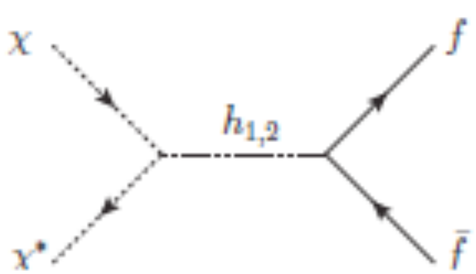
DM self-interaction

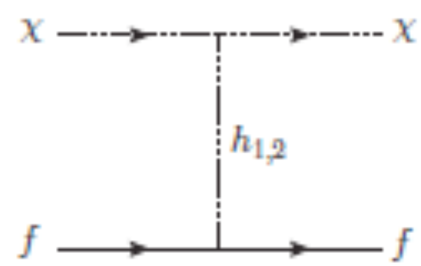


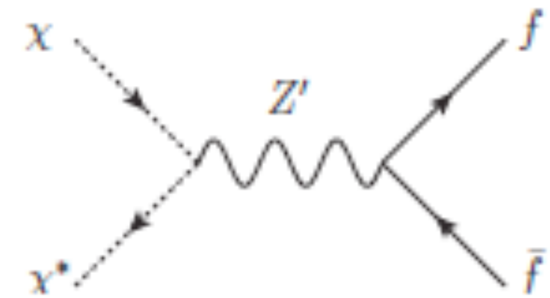
- SIMP relic density: $m_\chi = 0.03\,\alpha_{\rm eff}(T_{\rm eq}^2 M_P)^{1/3}$
- Bullet cluster & halo shape: $\sigma_{\rm self}/m_\chi \lesssim 1{\rm cm}^2/{\rm g}$
- Unitarity, perturbativity imposed.

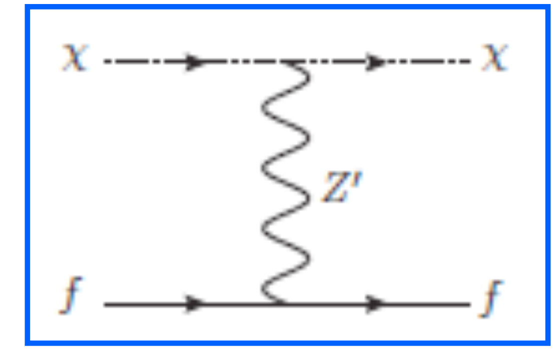
$$\lambda_\chi < 4\pi, \qquad |\mathcal{M}_{\chi\chi}| = \sqrt{2}\Big(2\lambda_\chi + 3R^2 + 4g_D^2 r^{-1}\Big) < 8\pi$$

Kinetic equil. condition









Higgs-portal:
lepton-Yukawa suppressed.

Z'-portal:
controlled by kinetic mixing.

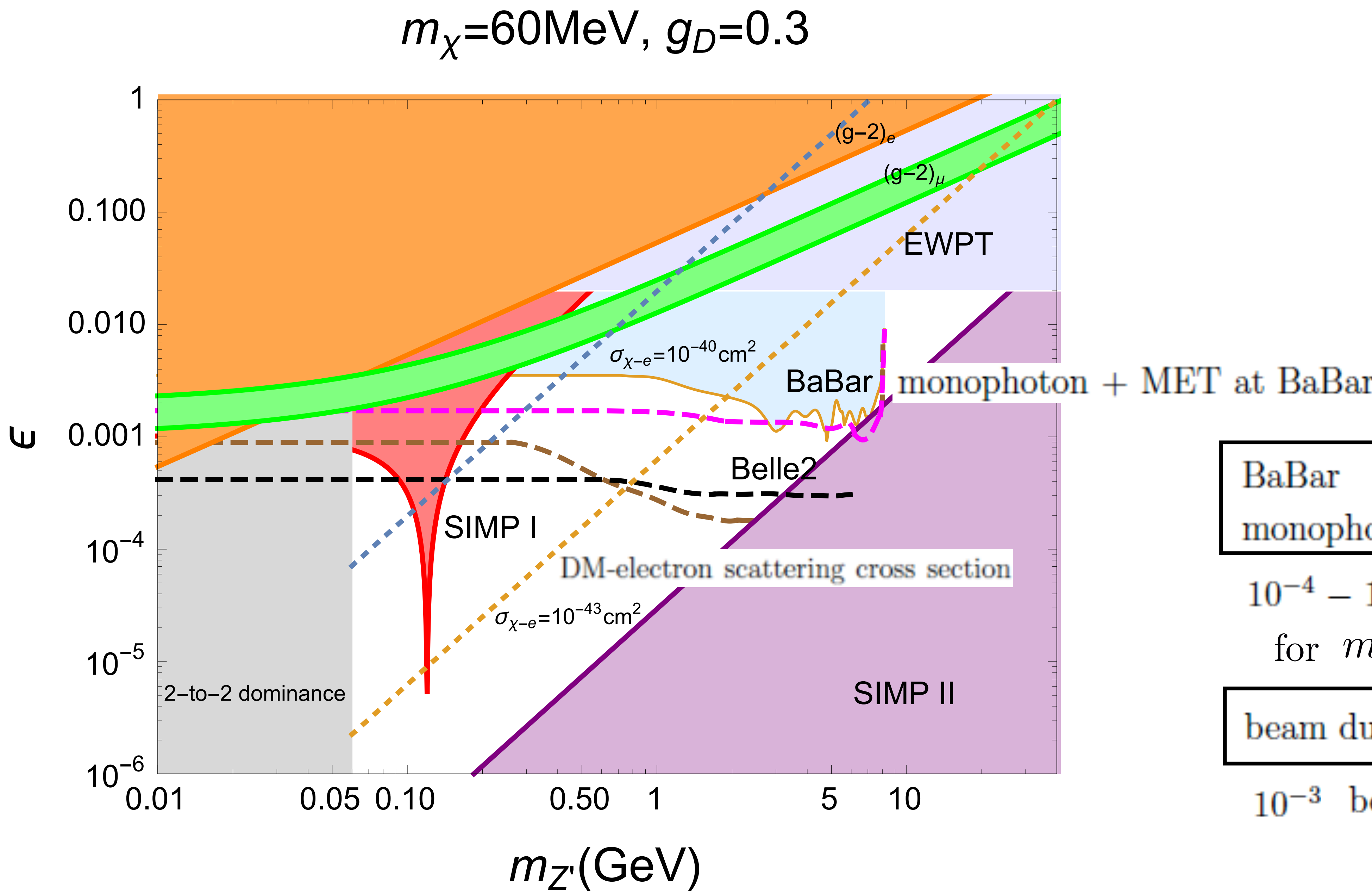
- SIMP conditions on 2→2 processes with Z' portal:

$$n_{\rm DM}\langle\sigma v_{\rm rel}\rangle_{\rm ann} < n^2_{\rm DM}\langle\sigma v^2_{\rm rel}\rangle_{3\to2} < n_{\rm SM}\langle\sigma v_{\rm rel}\rangle_{\rm kin}$$

$$\left.\begin{aligned}\langle\sigma v_{\rm rel}\rangle_{\rm ann} &= \frac{2\varepsilon^2 e^2 g_D^2 m_\chi^2}{\pi[(4m_\chi^2 - m_{Z'}^2)^2 + m_{Z'}^2\Gamma_{Z'}^2]}\left(\frac{T}{m_\chi}\right) \equiv \frac{\delta_1^2}{m_\chi^2} \\ \langle\sigma v_{\rm rel}\rangle_{\rm scatt} &= \frac{3\varepsilon^2 e^2 g_D^2 m_\chi^2}{2\pi m_{Z'}^4}\left(\frac{T}{m_\chi}\right) \equiv \frac{\delta_2^2}{m_\chi^2}.\end{aligned}\right\} \quad \begin{aligned}\delta_1 &\lesssim 2.4\times10^{-6}\,\alpha_{\rm eff}, \\ \delta_2 &\gtrsim 10^{-9}\alpha_{\rm eff}^{1/2}.\end{aligned}$$

$$\varepsilon \simeq \cos\theta_W\,\xi$$

SIMP & Z’ searches



BaBar
monophoton + dilepton

$10^{-4} - 10^{-3}$ $(\mathrm{BR}(l\bar{l}) = 1)$

for $m_{Z'} = 0.02 - 10.2$ GeV

beam dump (E137)

10^{-3} below $m_{Z'} = 0.1\,\mathrm{GeV}$.

- SIMP conditions are complementary for Z’ searches at colliders.

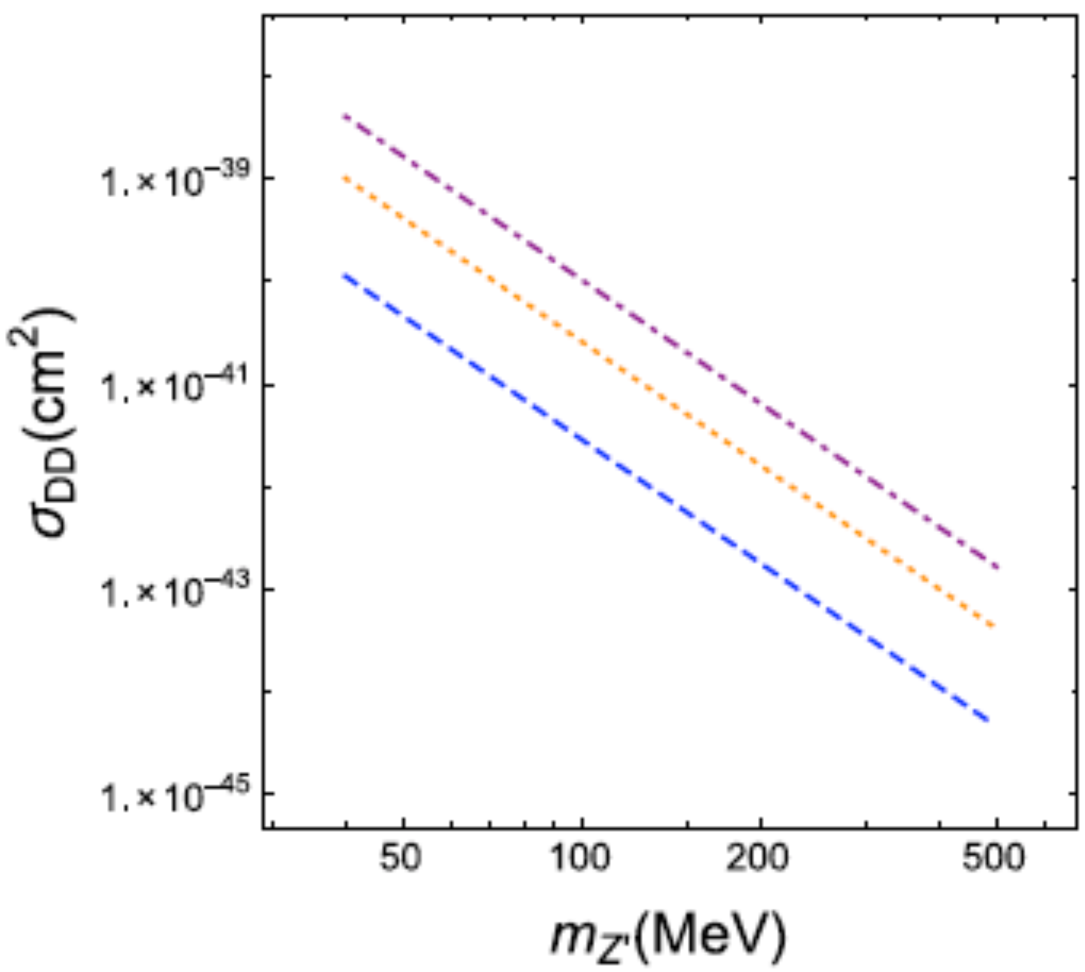
Direct detection

- SIMP-electron/nucleon elastic scattering releases nucleon/electron-recoil energy,

$$E = \frac{q^2}{2m_{e,N}} \sim 1 - 100\,\mathrm{eV}.$$

$(q = \mu\, v_{\rm DM},\ \mu = m_{e,N} m_\chi/(m_{e,N} + m_\chi).)$

Superconducting detectors?
[Hochberg et al (2015)]



$$\sigma_{\rm DD} = \frac{\varepsilon^2 e^2 g_D^2 \mu^2}{\pi m_{Z'}^4},$$

$$m_e, m_\chi, m_{Z'} \gg p \simeq m_\chi v_{\rm DM}$$

⇒ independent of SIMP mass.

XENON10:

$$\sigma_{\rm DD} < 2 \times 10^{-36}\,\mathrm{cm}^2$$

at best around $m_\chi = 30\,\mathrm{MeV}$.

Conclusions

- SIMP paradigm leads to testable scenarios via DM self-interactions as well as possibly, messengers particles.
- SIMP dark mesons can be in kinetic equilibrium with Z' portal, remaining stable.
- Scalar SIMP dark matter with gauged Z_3 has a built-in Z'-portal.
- SIMP conditions are complementary to Z' searches at colliders.

Backup

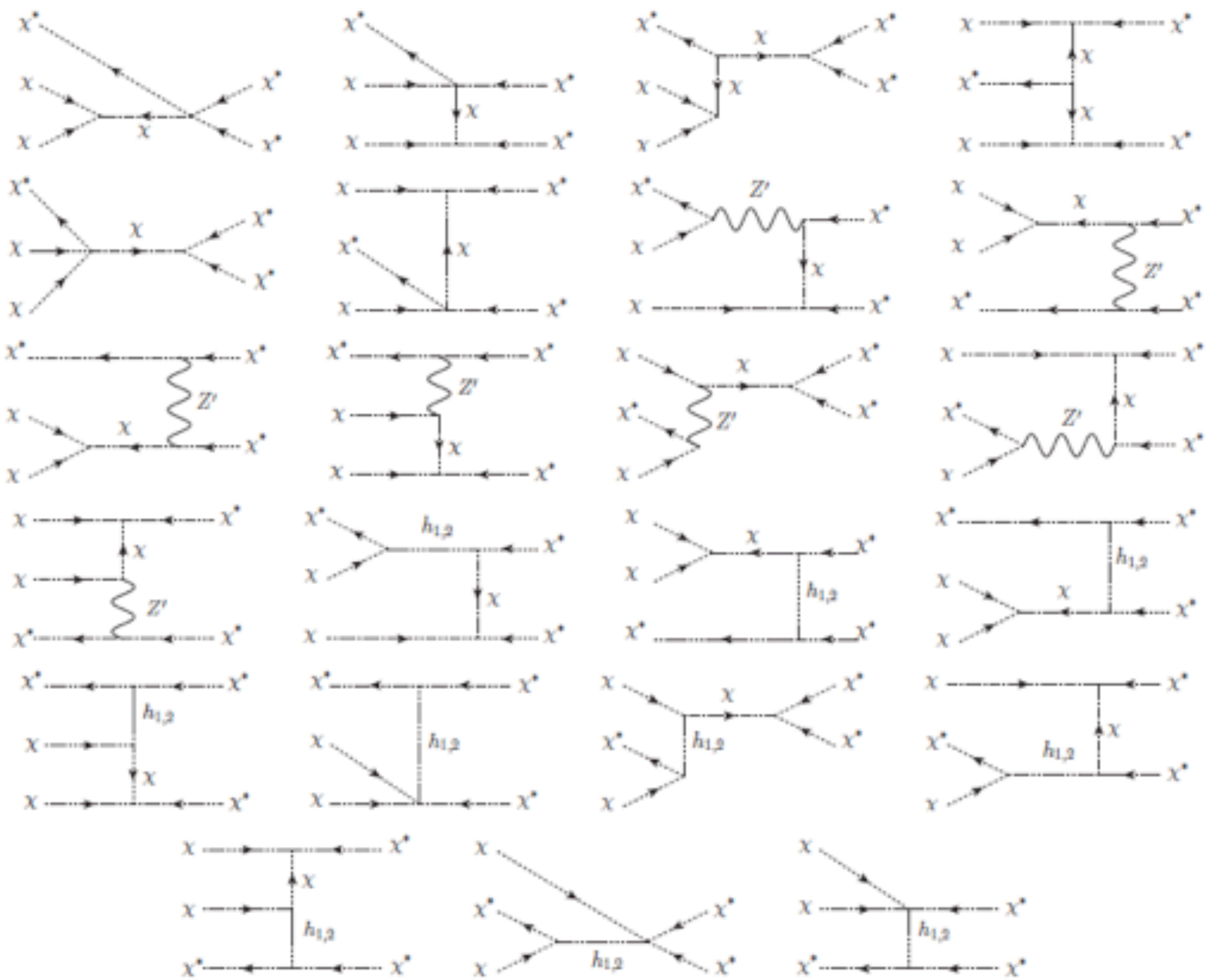


Figure 1: Feynman diagrams for $\chi\chi\chi^* \to \chi^*\chi^*$.

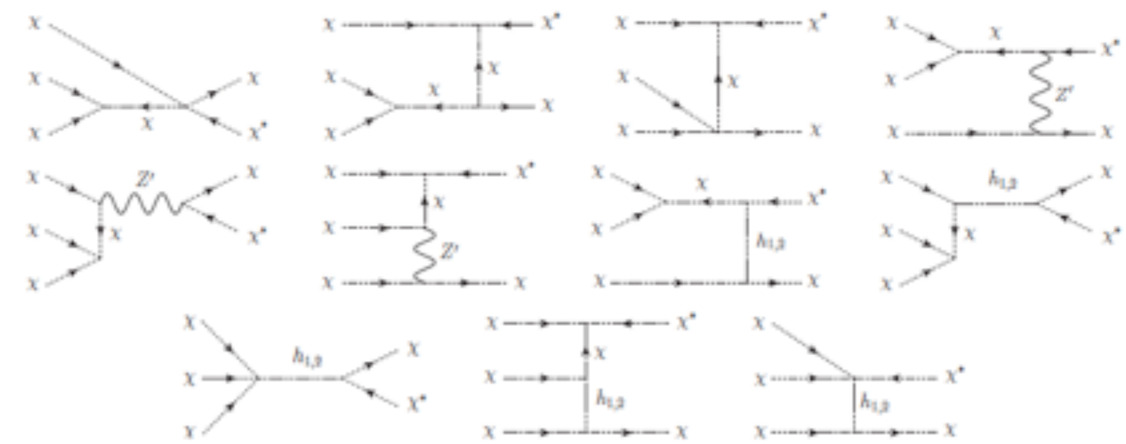


Figure 2: Feynman diagrams for $\chi\chi\chi \to \chi\chi^*$.

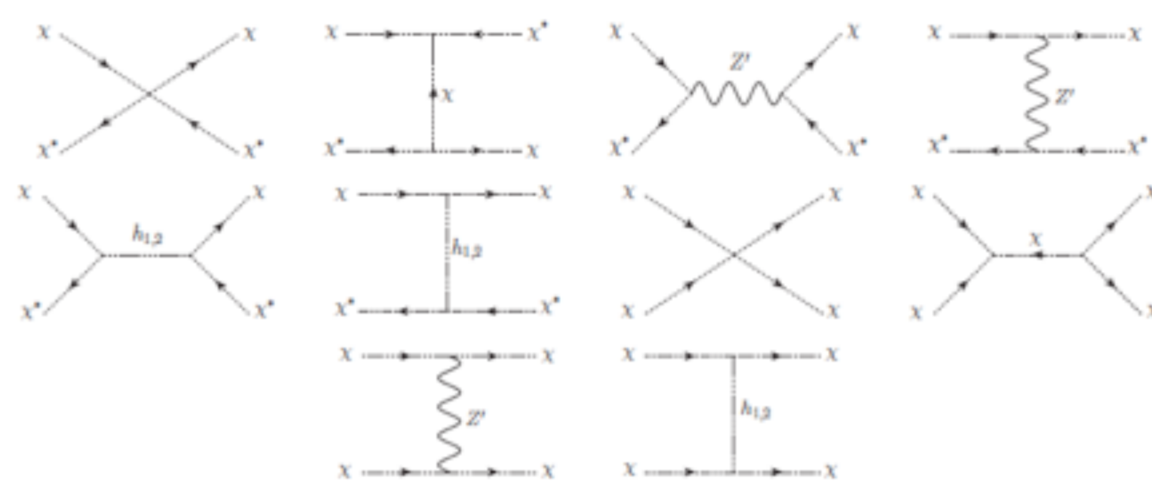


Figure 3: Feynman diagrams for $\chi\chi^* \to \chi\chi^*$ and $\chi\chi \to \chi\chi$.