Nonparametric test of LCDM model using Planck temperature and polarization angular power spectrum data

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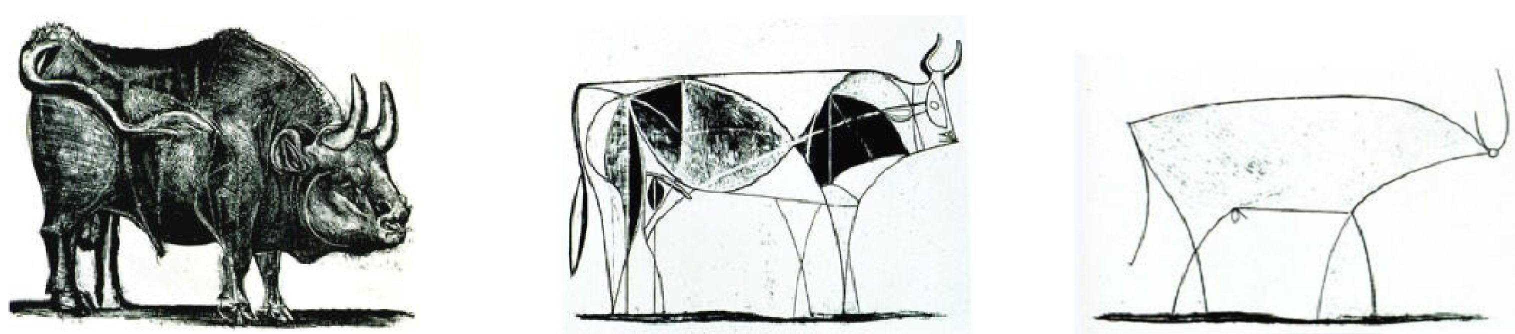


Bull pictures courtesy of “Pablo Picasso”

All models are false, some are useful. (George E. P. Box)

I will talk about:

- Model-independent analysis of CMB temperature and polarization angular power spectra

  in collaboration with:

  Arman Shafieloo (KASI, South Korea)

  Mihir Arjunwadkar(CMS, India)

  Tarun Souradeep(IUCAA, India)

- Using Gaussian Process for consistency test between best-fit LCDM model and CMB angular power spectrum data

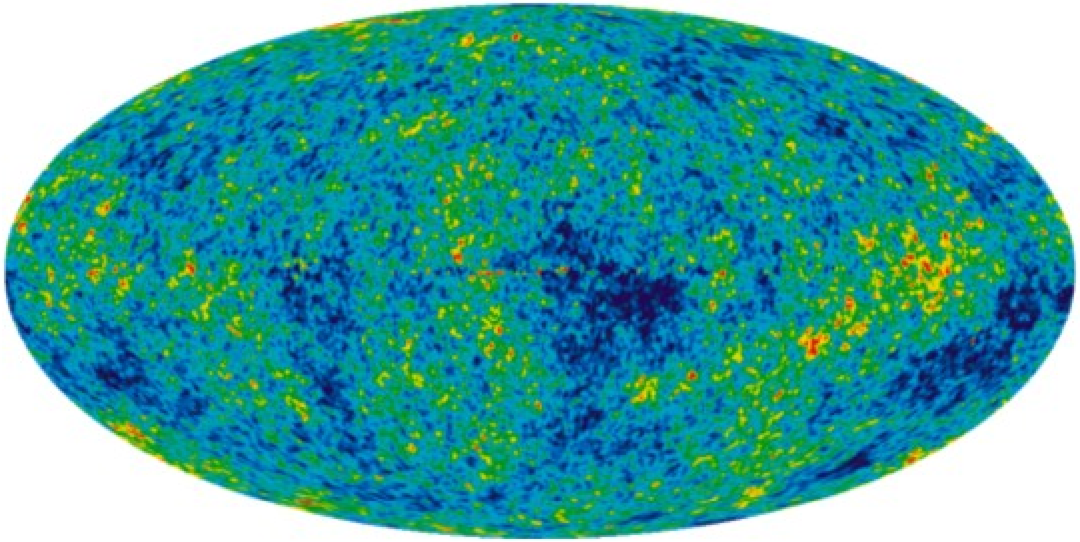
  in collaboration with:

  Arman Shafieloo (KASI, South Korea)

  Jan Hamann (University of Sydney, Australia)

Model-independent analysis of CMB temperature and polarization angular power spectra

CMB Anisotropies and the Power Spectrum



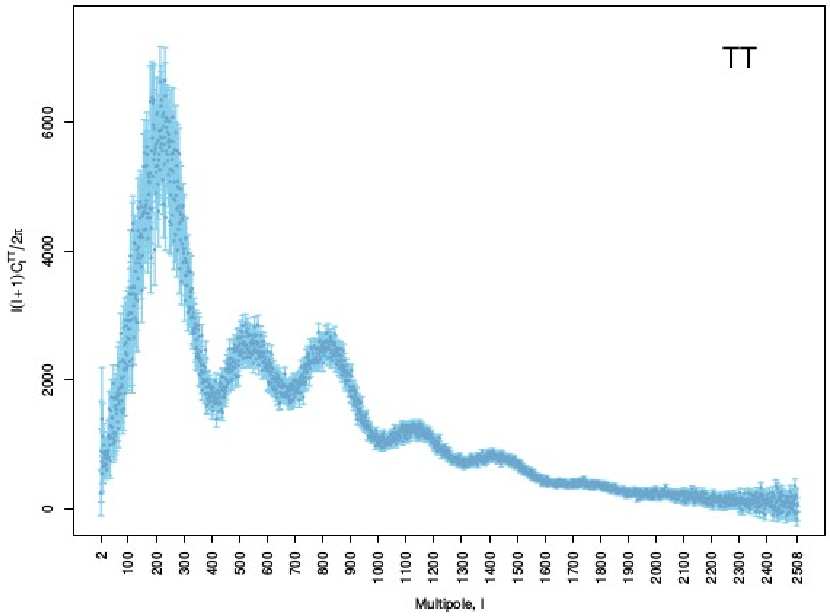
- **Expansion in spherical harmonics**

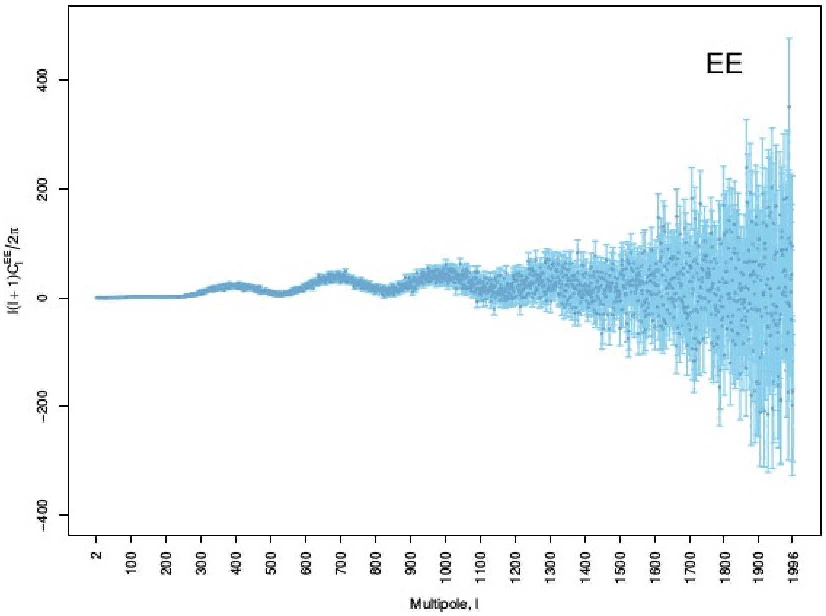
$$\Delta T(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} a_{l,m} Y_{l,m}(\theta, \phi),$$

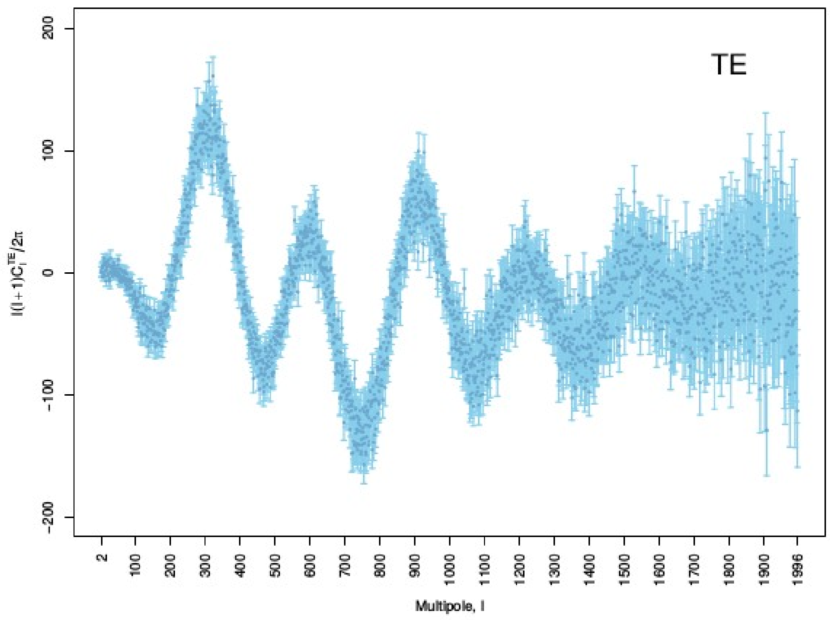
- **$\Delta T(\theta, \phi)$ is a Gaussian random field**
  $\longrightarrow$ $a_{l,m}$ are mean-0 random variables with variance $C_l := E|a_{l,m}|^2$.

$$\tilde{C}_l := \frac{1}{2l+1} \sum_{m=-l}^{+l} |a_{l,m}|^2$$

Planck 2015: Power Spectrum Data





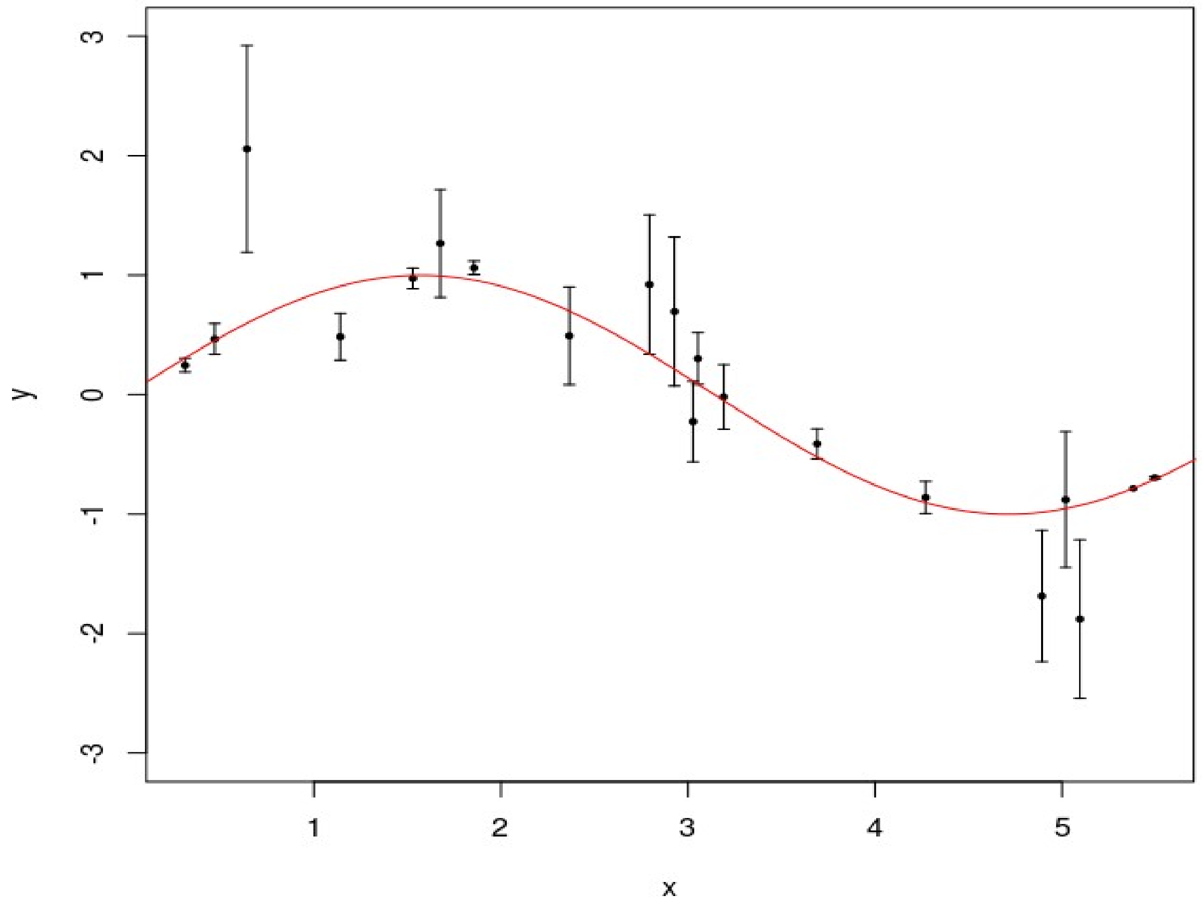


(http://pla.esac.esa.int/pla/cosmology)

Regression Problem

Data: $$(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$$

$$Y = f(x) + \epsilon$$



Regression Problem

- **Canonical form of the regression problem:** Given data $(X_1, Y_1), \ldots, (X_n, Y_n)$ where $Y_i = f(x_i) + \epsilon_i$ estimate relationship f between X and Y.
- **Formally:** $f(x) = E(Y|X = x)$.
- **Hitch:** Typically, there is only one realization of Y at each x.
- **Solution:** Make reasonable assumptions about the noise ϵ_i, try to make a reasonable guess $\widehat{f}(x)$.
- **Parametric/model-based regression:** Assume a specific functional form for $\widehat{f}(x)$ with *finite* number of adjustable parameters. Estimate adjustable parameters by maximizing the likelihood function.

Parametric regression

$Y_i = f(x_i) + \epsilon_i$

- Assume $f(x) = ax + b$.
- Assume noise $\epsilon_i \sim N(0, \sigma^2)$ IID.
- Likelihood function

$$L(a, b|\text{data}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(Y_i - (ax_i + b))^2}{2\sigma^2}\right)$$

- To estimate a, b: Maximize $L(a, b|\text{data})$ w.r.t. a, b.
- This is same as linear least-squares regression, under the assumptions made.

Regression Problem

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- **Parametric/model-based regression:** Assume a specific functional form for $\widehat{f}(x)$ with *finite* number of adjustable parameters. Estimate adjustable parameters by maximizing the likelihood function.
- **Nonparametric/model-independent regression:** No functional form assumed; let the data drive the fit through optimal smoothing. Technically, infinite number of parameters.

REACT: nonparametric regression

- $Y_i = f(x_i) + \epsilon_i$, with $\epsilon_i \sim N(0, \sigma^2)$ IID, σ^2 known.
- Assume $f \in L_2(a, b)$ and a complete orthonormal basis $\{\phi_j(x)\}$.

$$f(x) = \sum_{j=0}^{\infty} \beta_j \phi_j(x), \;\; \beta_j = \int_a^b f(x)\phi_j(x)dx$$

- Regression estimator $\widehat{f}(x)$:

$$f(x) = \sum_{j=0}^{n-1} \widehat{\beta}_j \phi_j(x) + \text{(some truncation bias)}$$

$\widehat{\beta}_j := \lambda_j Z_j$ with $1 \geq \lambda_0 \geq \ldots \geq \lambda_{n-1} \geq 0.$ and $Z_j = \sum_{i=1}^{n} Y_i \phi_j(x_i)$

- Inverse-noise-weighted squared loss function

$$L(\widehat{f}, f) = \int \left(\frac{\widehat{f}(x) - f(x)}{\sigma(x)} \right)^2 dx.$$

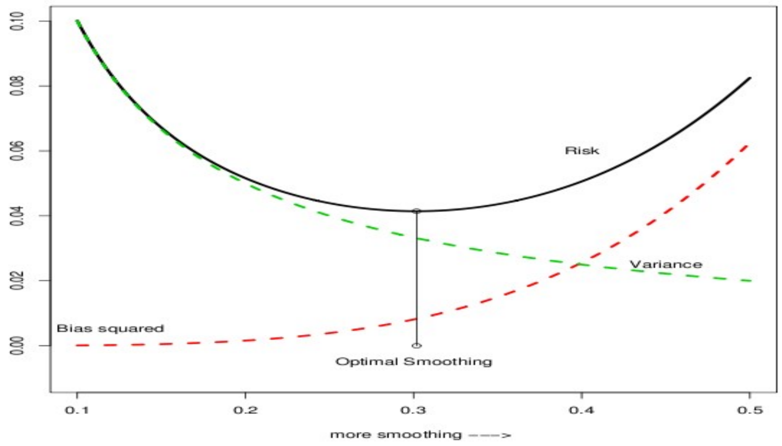
- Risk estimator

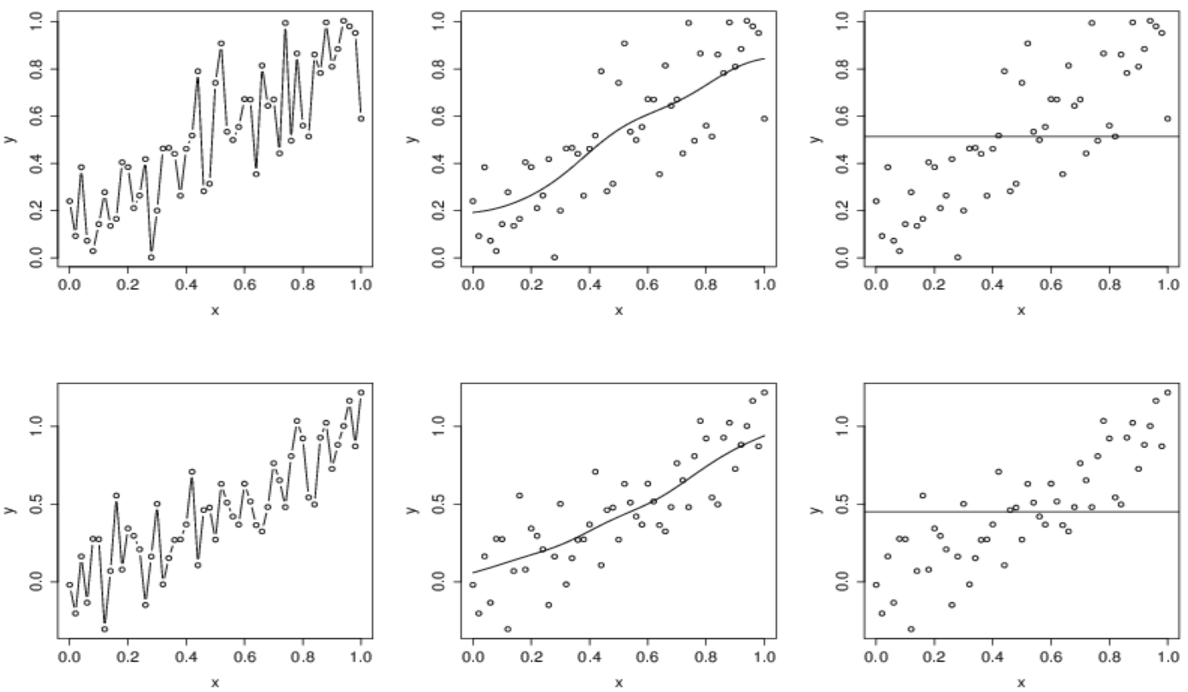
$$\widehat{R}(\lambda) = Z^T \bar{D} W \bar{D} Z + \text{tr}(DWDB) - \text{tr}(\bar{D} W \bar{D} B),$$

subject to the constraint $1 \geq \lambda_0 \geq \ldots \geq \lambda_{n-1} \geq 0.$

Beran 1996
Bean & Dumbgen 1998
Beran 2000
Genovese et al. 2004
Bryan et al 2007
Aghamousa et al. 2012
Aghamousa et al. 2014
Aghamousa et al. 2015
Aghamousa & Shafieloo 2015

Bias-Variance trade-off





REACT: confidence ball

Beran and Dümbgen (1998)

- The REACT formalism also gives a powerful inferential entity: the REACT confidence set around the fit $\widehat{f}$.
- $$B = \left\{\beta \in R^n : ||\beta - \widehat{\beta}||^2 \leq \rho^2\right\}$$

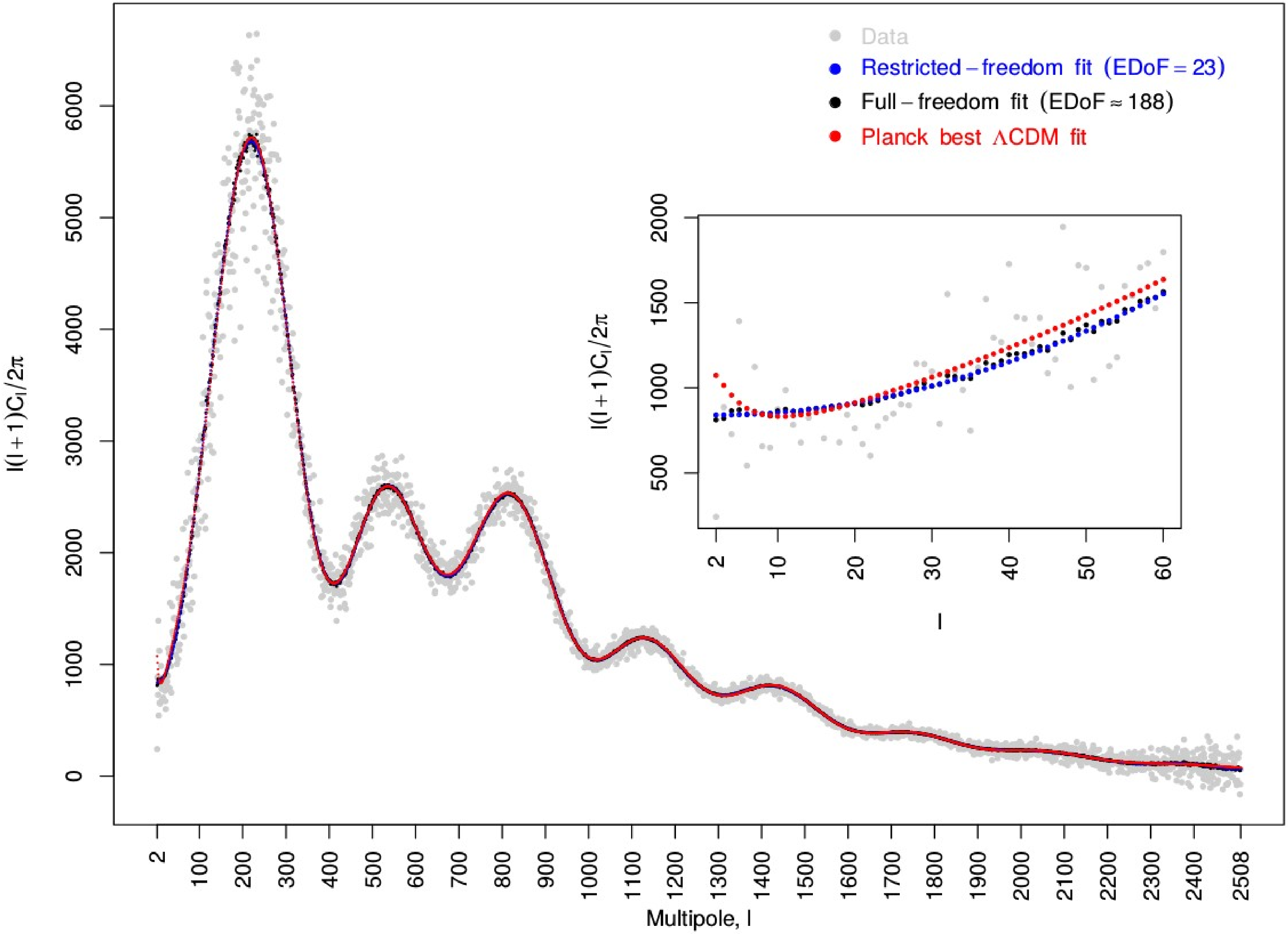
  $$\rho^2 = R(\lambda_*) + \widehat{\tau} z_\alpha / \sqrt{n}$$

  $$\widehat{\tau}^2 = 2\sigma^2 \left(\sigma^2 + 2\sum_J^{n-1}\left(Z_j^2 - \frac{\sigma^2}{n}\right)\right)$$
- This is a spherical object centered around the fit $\widehat{f}$ in the L_2 space such that

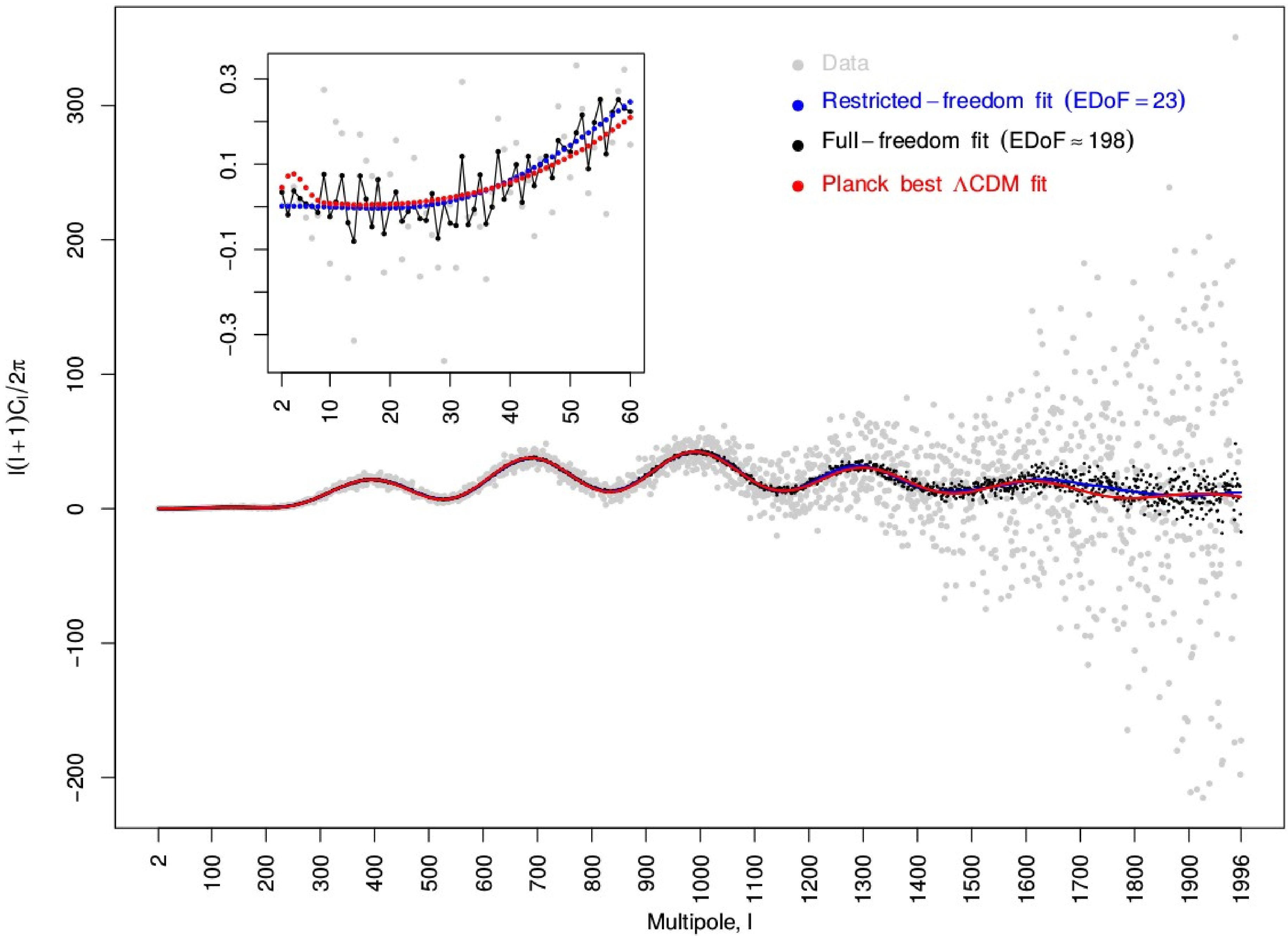
  $$P(\beta \in B) \longrightarrow 1 - \alpha$$

  (asymptotically)

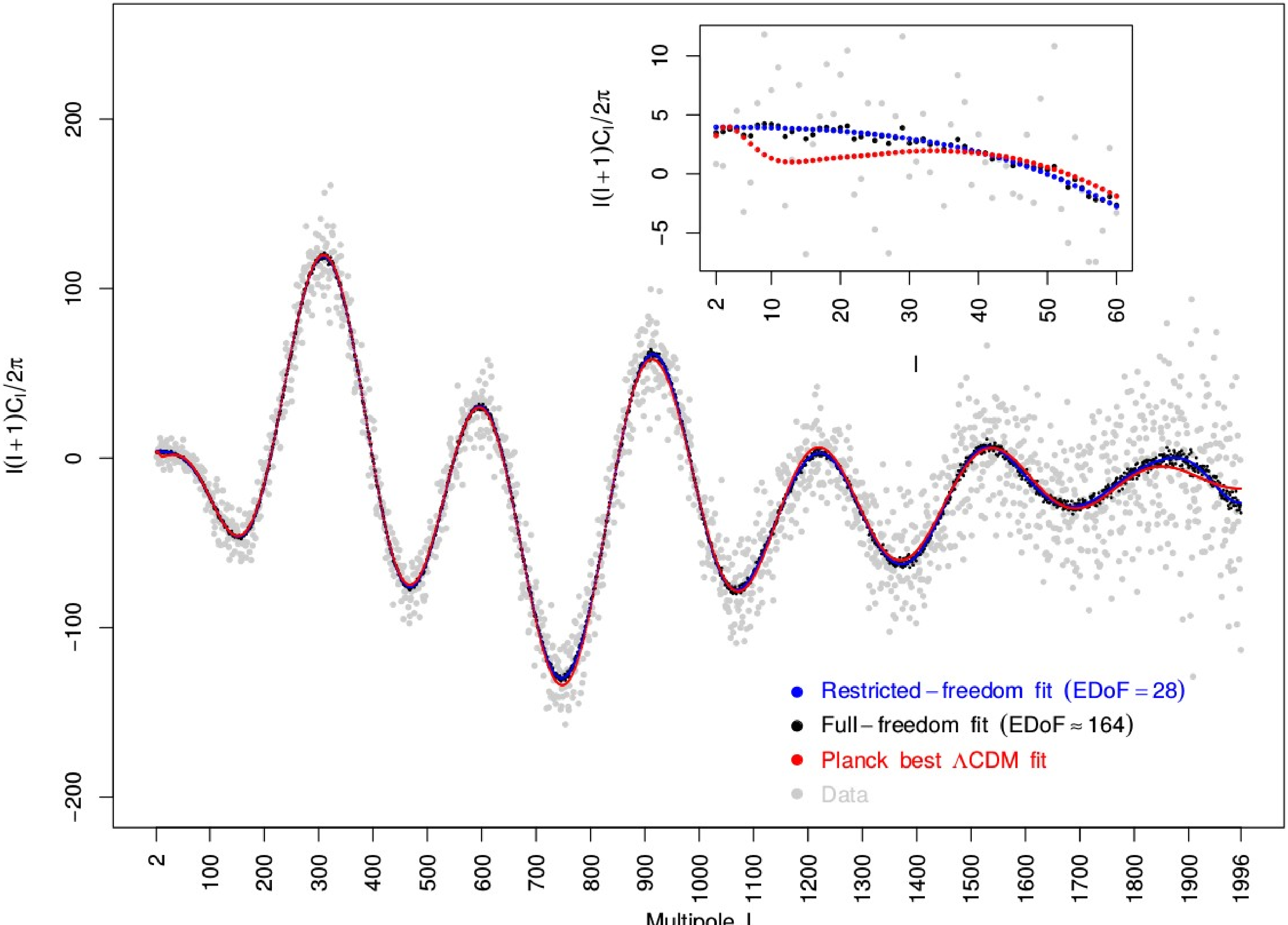
TT nonparametric fits



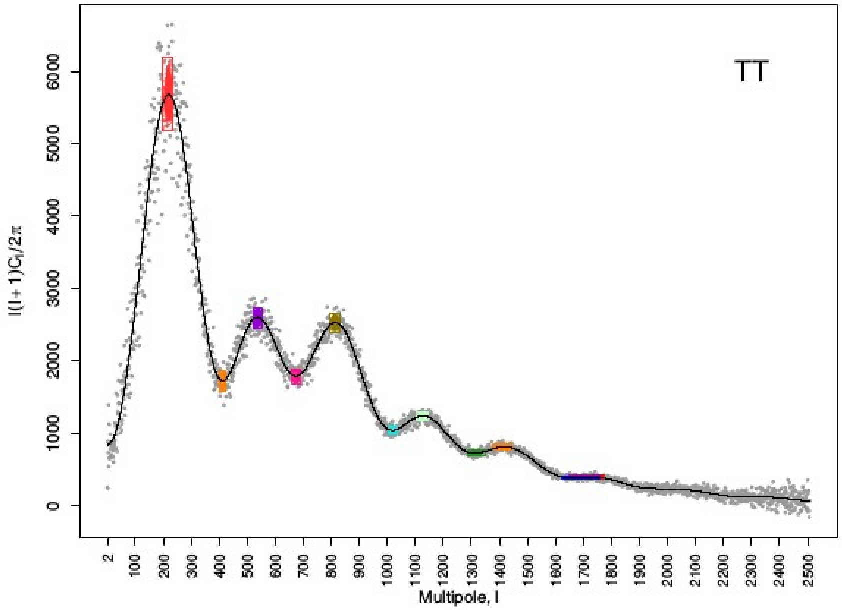
EE nonparametric fits

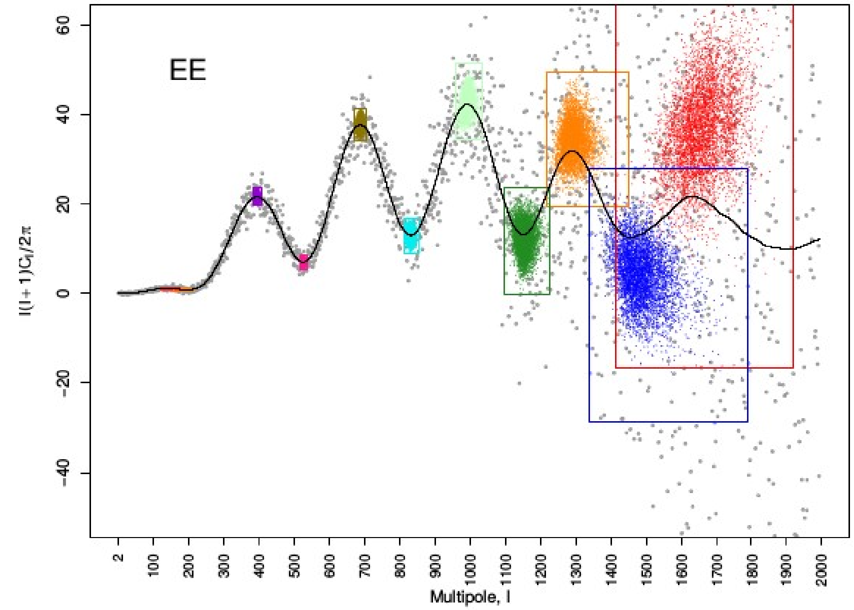


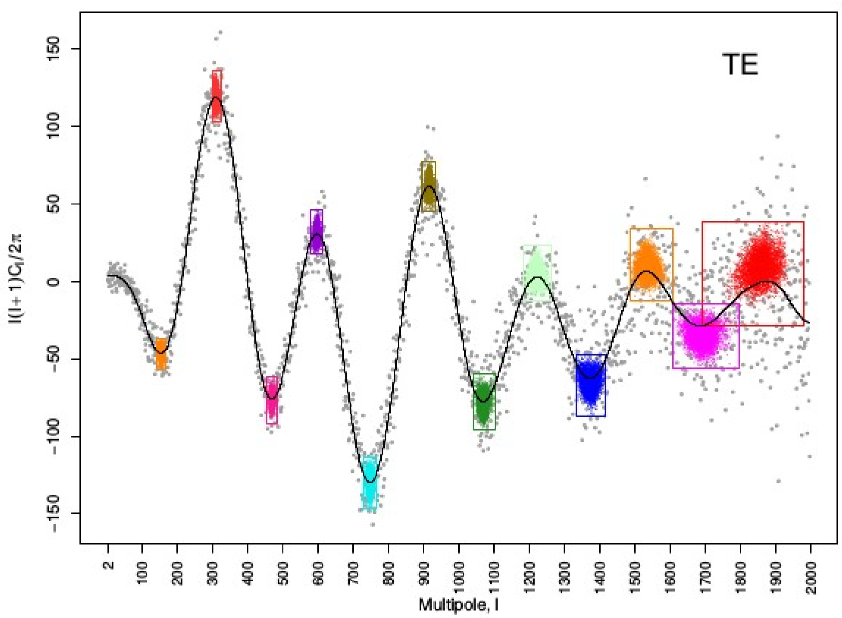
TE nonparametric fits



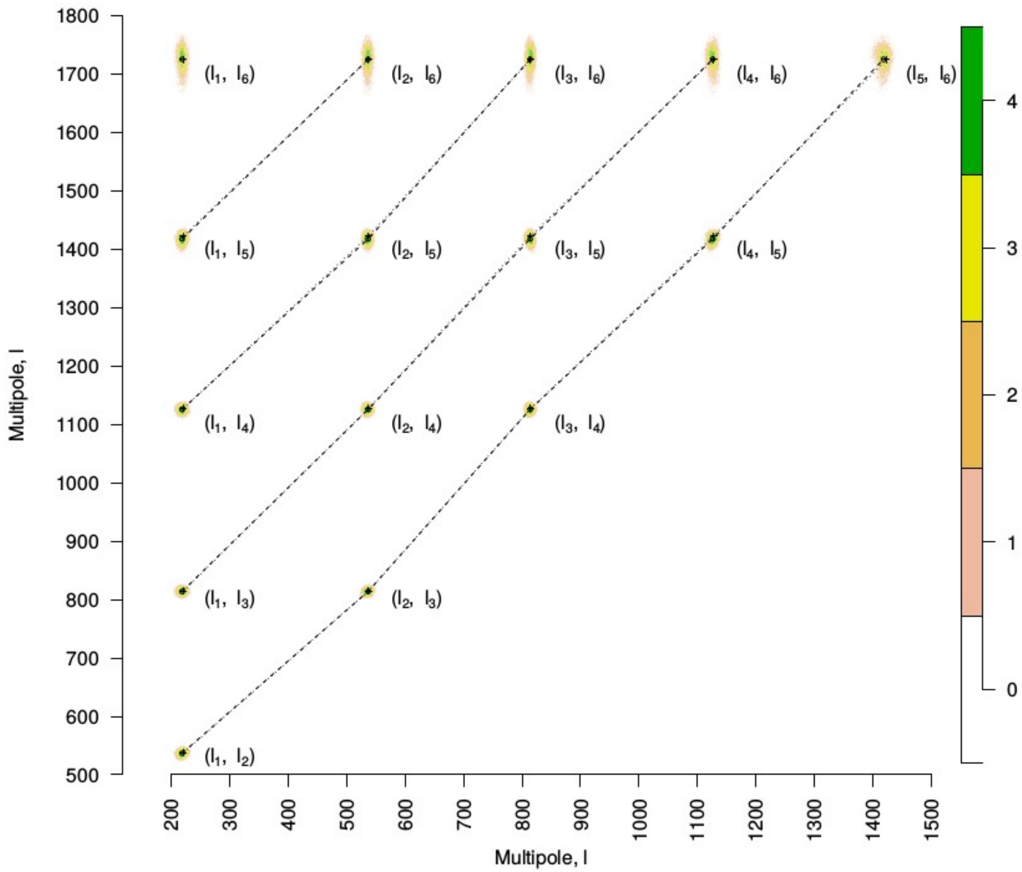
Uncertainties on peaks and dips



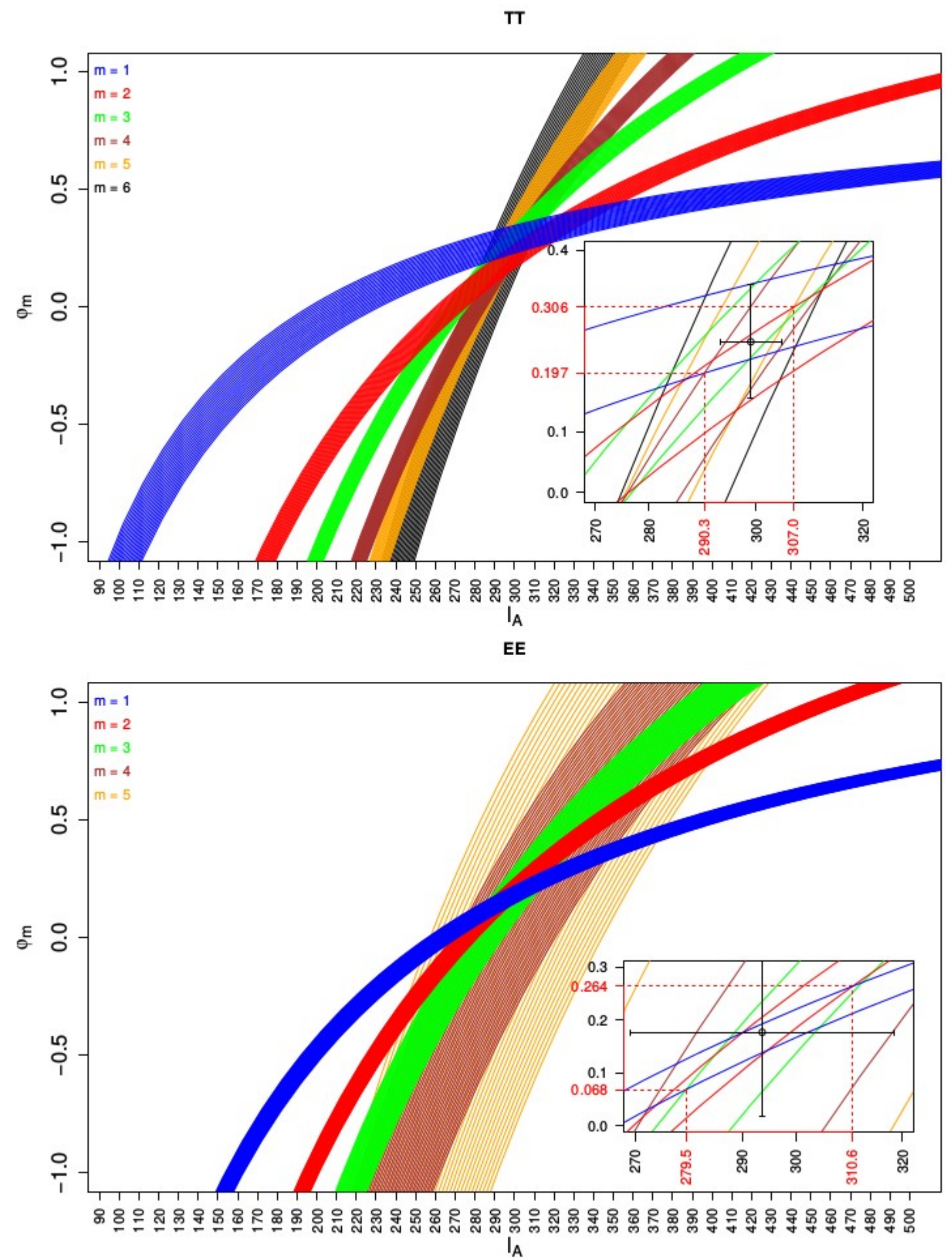




Harmonicity of acoustic oscillations

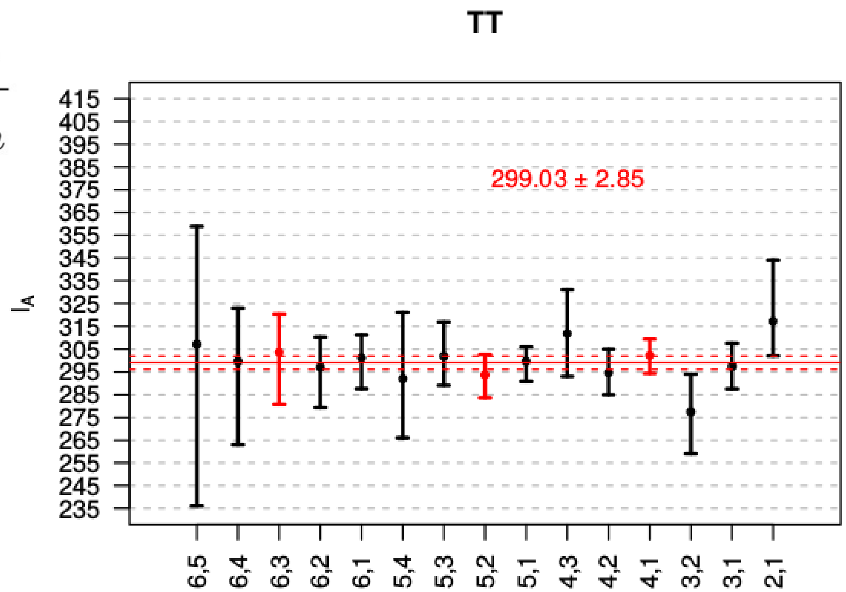


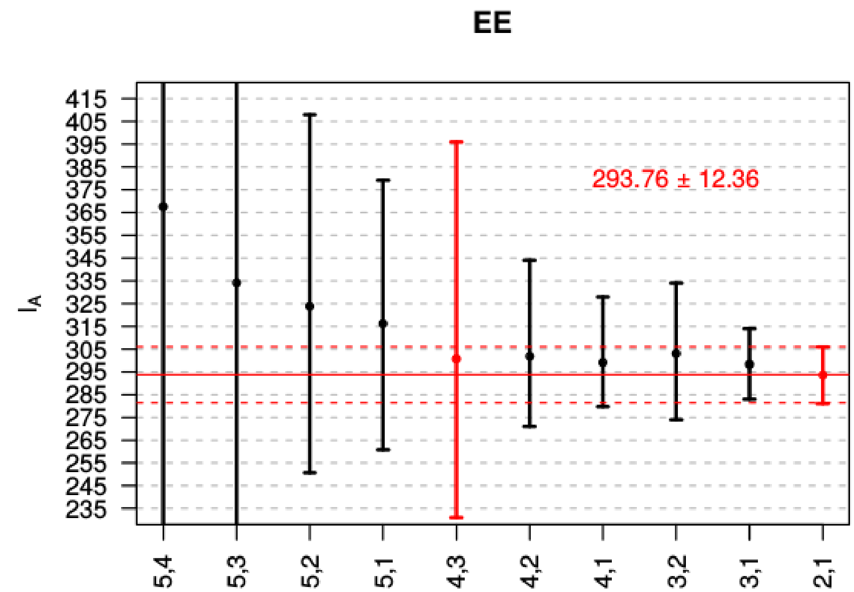
$$\ell_m = \ell_A(m - \phi + \Delta m)$$



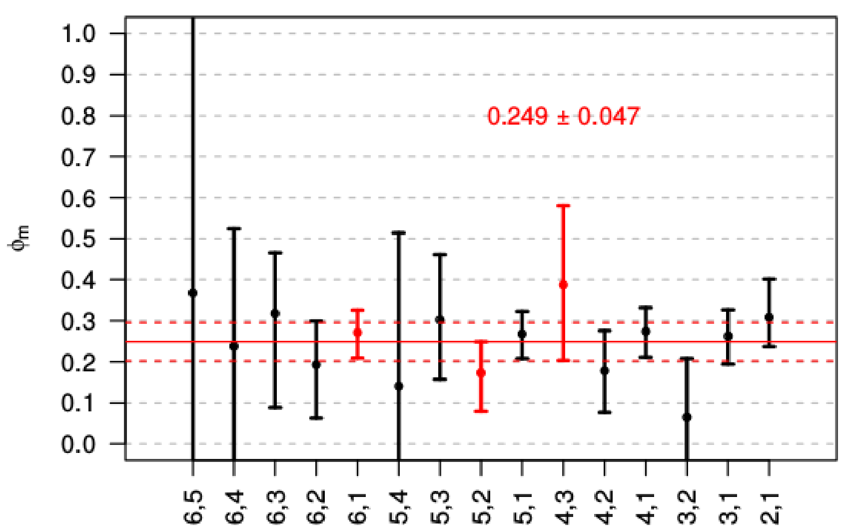
Acoustic scale and phase shift

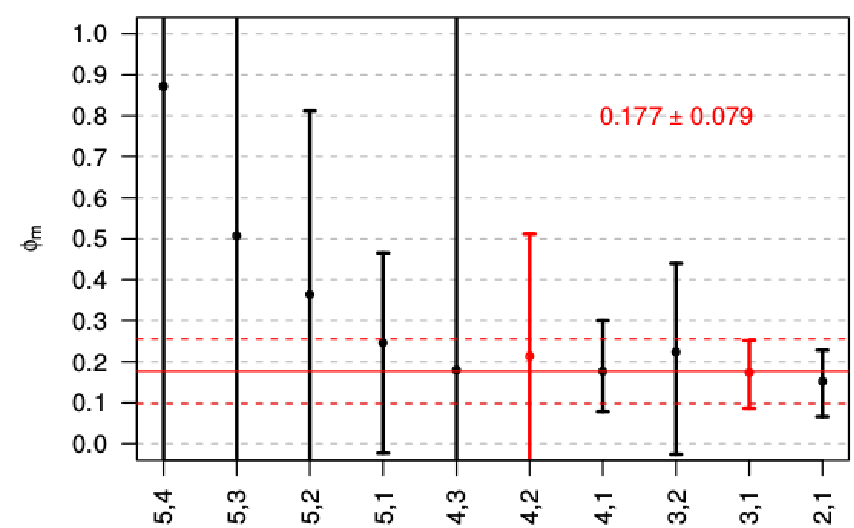
$$\ell_A = \frac{m-n}{l_m - l_n}$$





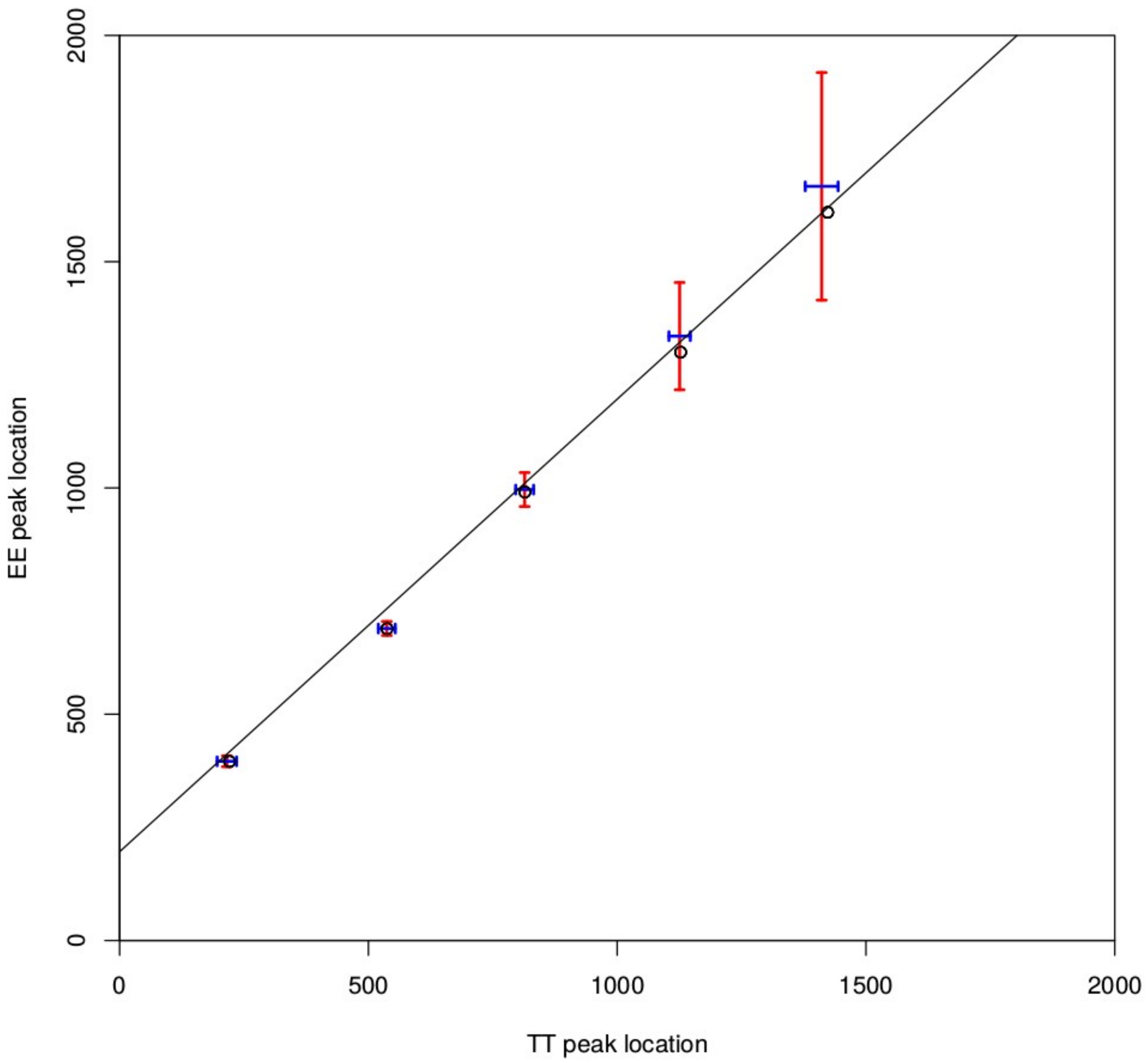
$$\phi = m - \frac{\ell_m}{\ell_A} + \Delta m$$





TT-EE phase shift

$$\ell_m^{EE} = \ell_m^{TT} + (\Delta m)^{EE} \times \ell_A$$



$$l_A \approx 300 \text{ leads to } (\Delta m)^{EE} \approx 0.65$$

Gaussian Process on CMB angular power spectrum data

Gaussian Processes

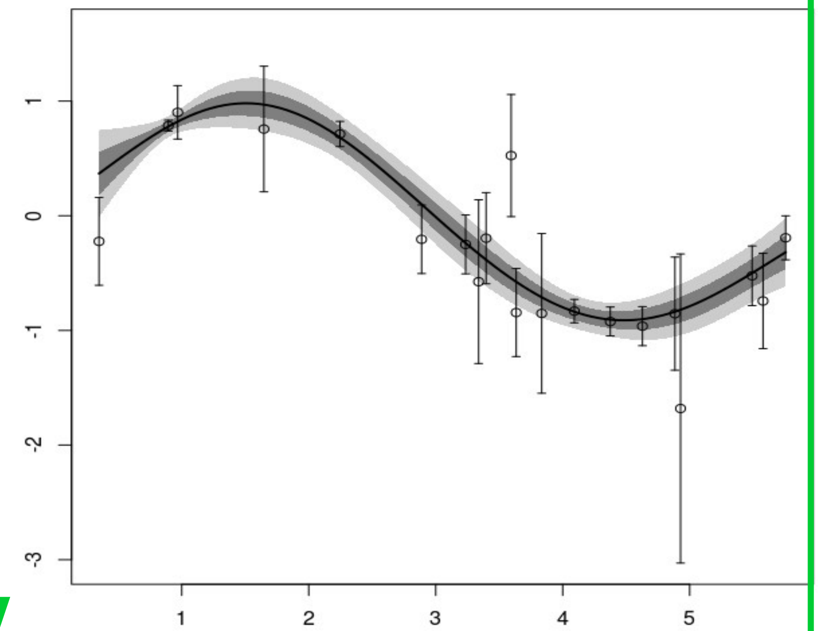
$$y \sim N(\mu, \quad K(X,X) + C)$$

Squared Exponential Covariance

$$k(x_i, x_j) = \sigma^2 \exp\left(-\frac{(x_i - x_j)^2}{2l^2}\right)$$

$$[K(X,X)]_{ij} = k(x_i, x_j)$$

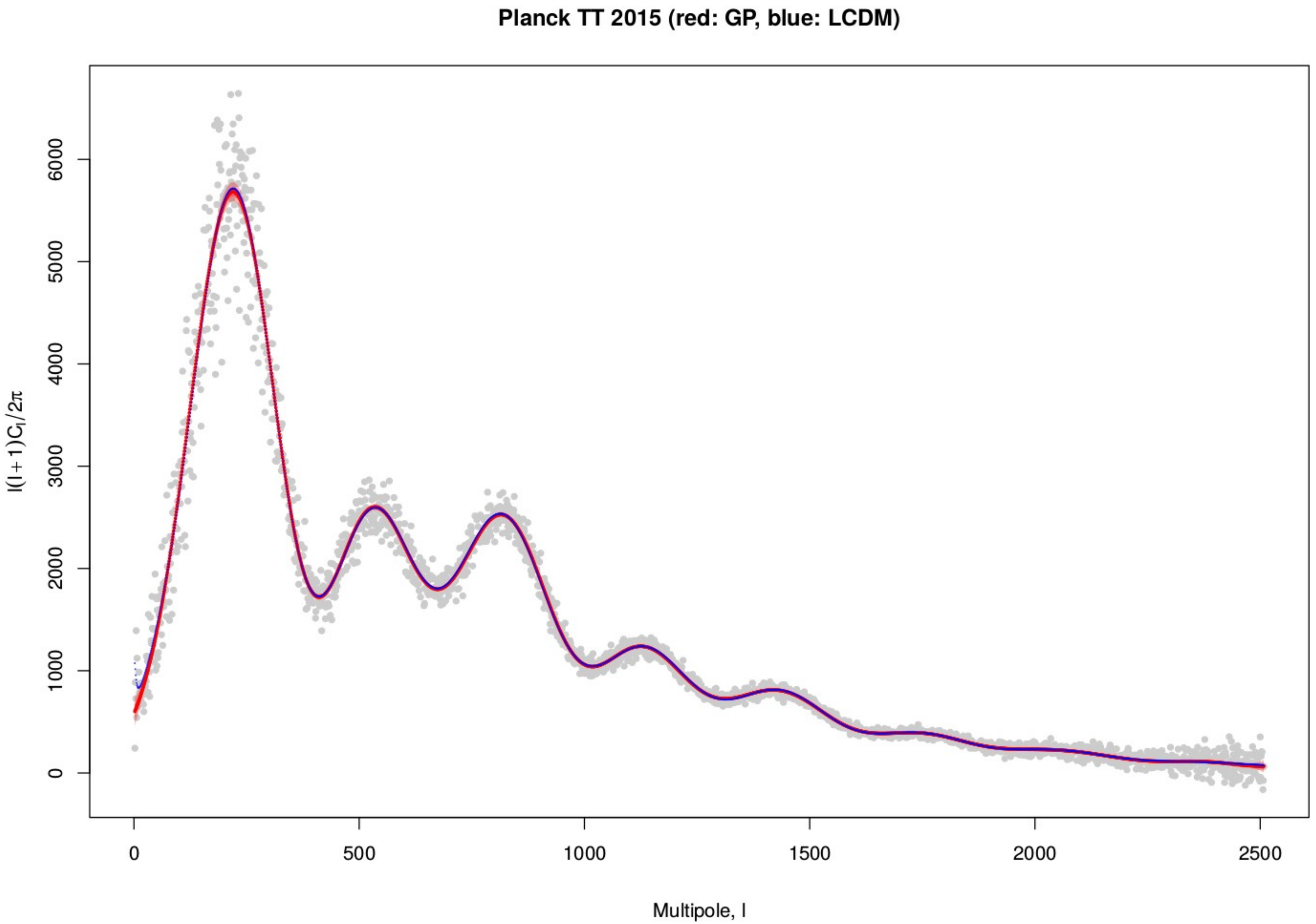
$$\begin{bmatrix} y \\ f^* \end{bmatrix} = N\left(\begin{bmatrix} \mu \\ \mu^* \end{bmatrix}, \begin{bmatrix} K(X,X)+C & K(X,X^*) \\ K(X^*,X) & K(X^*,X^*) \end{bmatrix}\right)$$



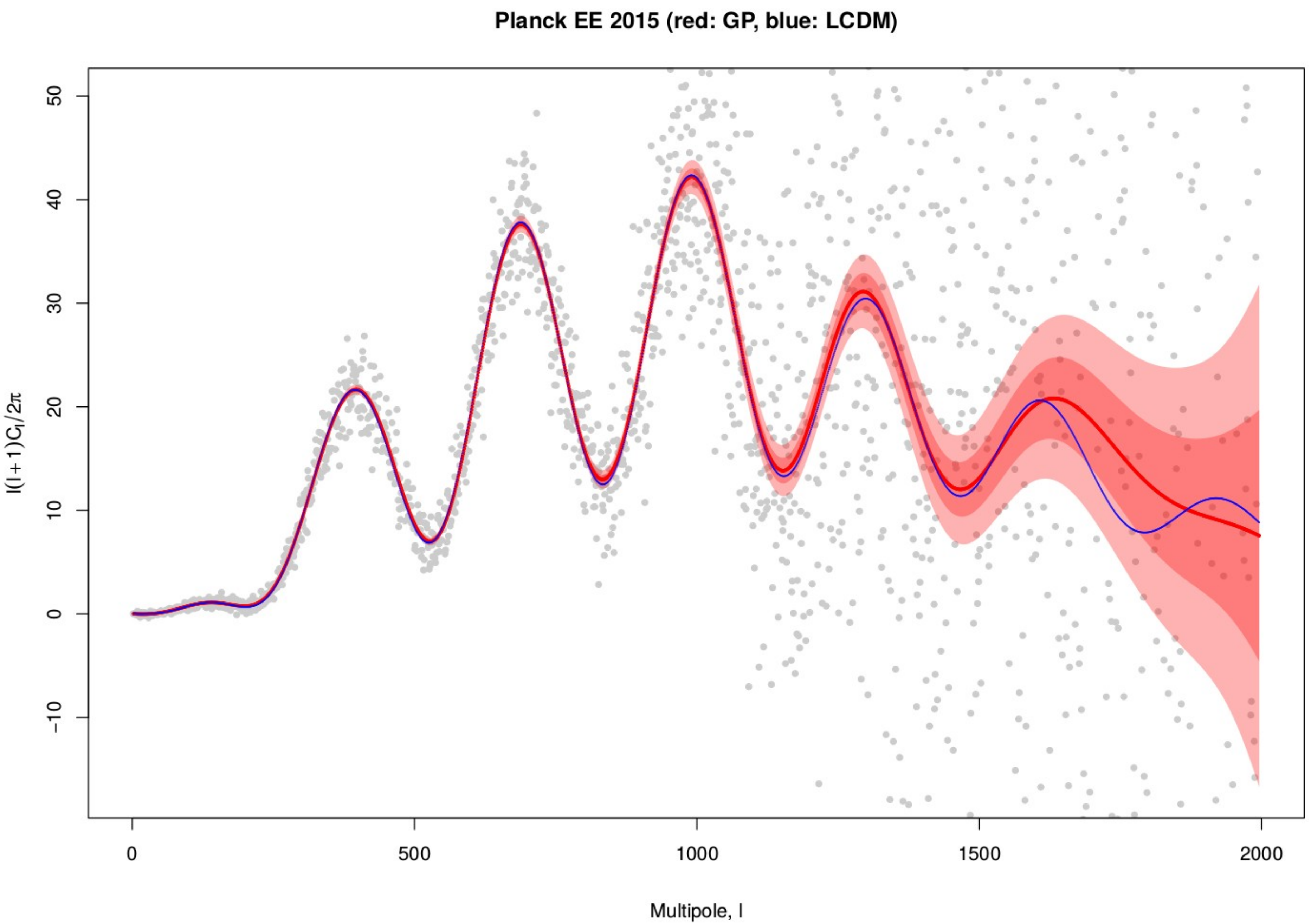
$$\overline{f^*} = \mu^* + K(X^*,X)[K(X,X)+C)]^{-1}(y-\mu)$$

$$\mathrm{Cov}(f^*) = K(X^*,X^*) - K(X^*,X)[K(X,X)+C]^{-1}K(X,X^*)$$

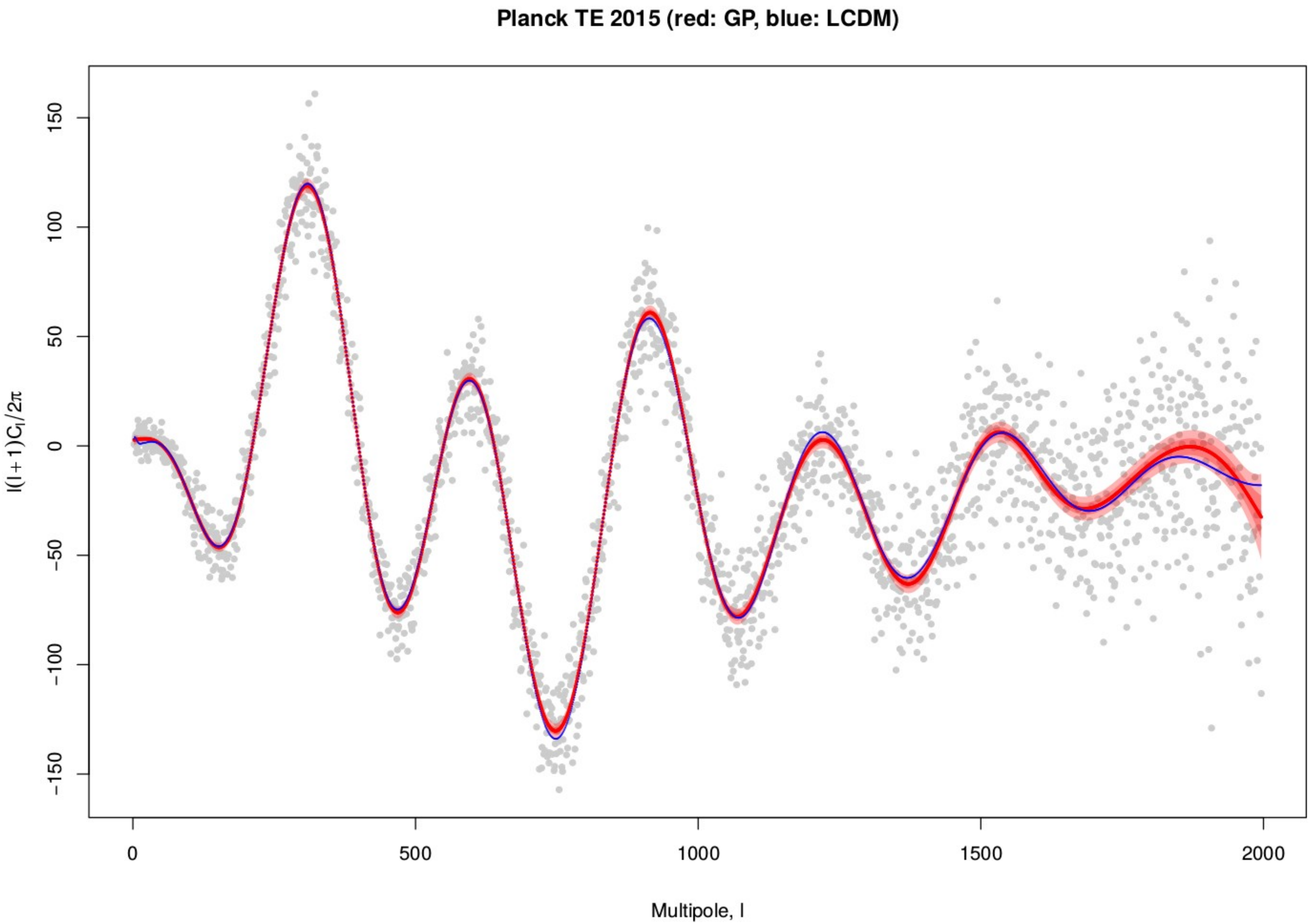
$$\log(\mathfrak{L}) = -\frac{1}{2}(y-\mu)^T[K(X,X)+C]^{-1}(y-\mu) - \frac{1}{2}\log|K(X,X)+C| - \frac{n}{2}\log 2\pi$$



Planck TT 2015 (red: GP, blue: LCDM)
0
1000
2000
3000
4000
5000
6000
l(l+1)C_l/2π
0
500
1000
1500
2000
2500
Multipole, l



Planck EE 2015 (red: GP, blue: LCDM)
50
40
30
20
10
0
−10
l(l+1)C_l/2π
0
500
1000
1500
2000
Multipole, l



Planck TE 2015 (red: GP, blue: LCDM)
l(l+1)C_l/2π
Multipole, l
0
500
1000
1500
2000
−150
−100
−50
0
50
100
150

Gaussian Processes

$$y \sim N(\mu, \quad K(X,X) + C)$$

Squared Exponential Covariance

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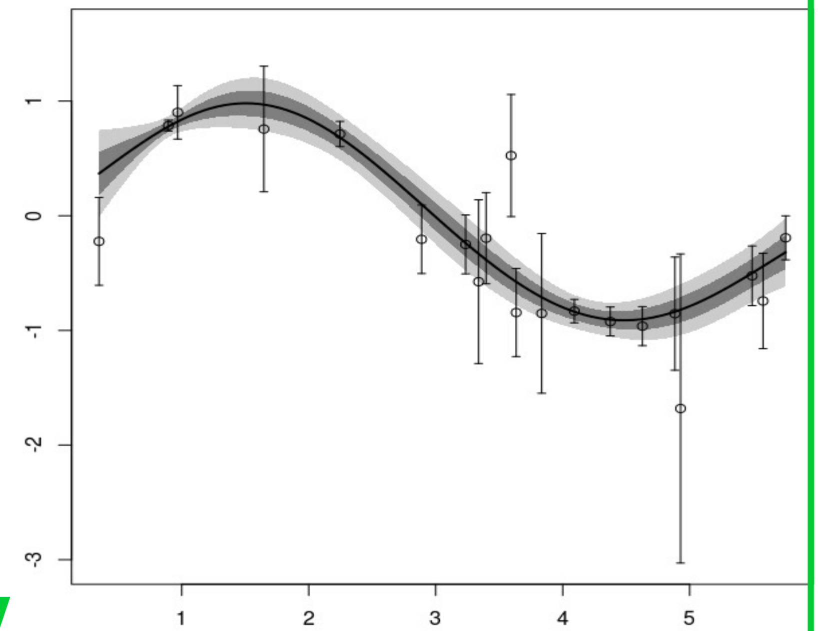
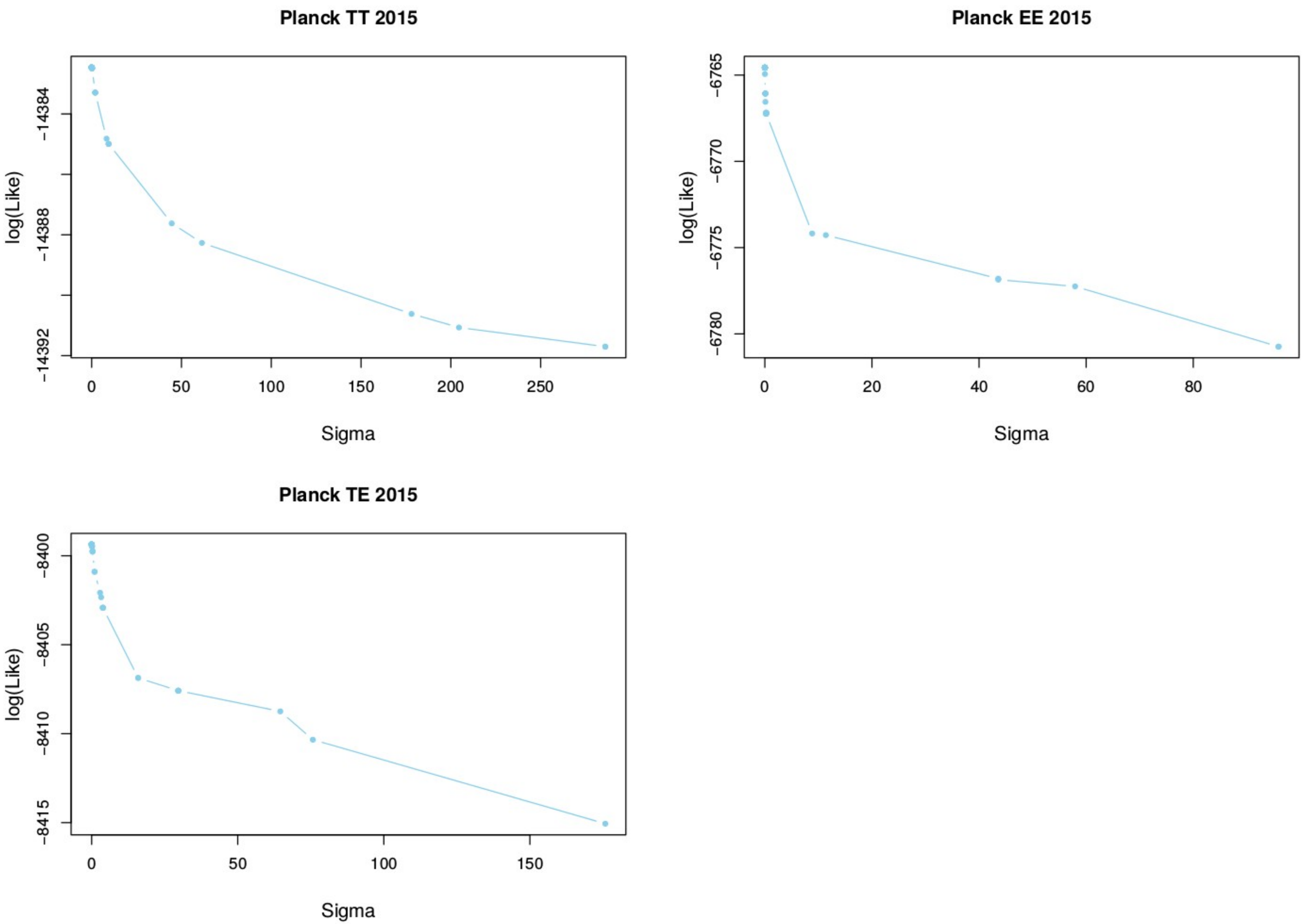


$$\overline{f^*} = \mu^* + K(X^*,X)[K(X,X)+C)]^{-1}(y-\mu)$$

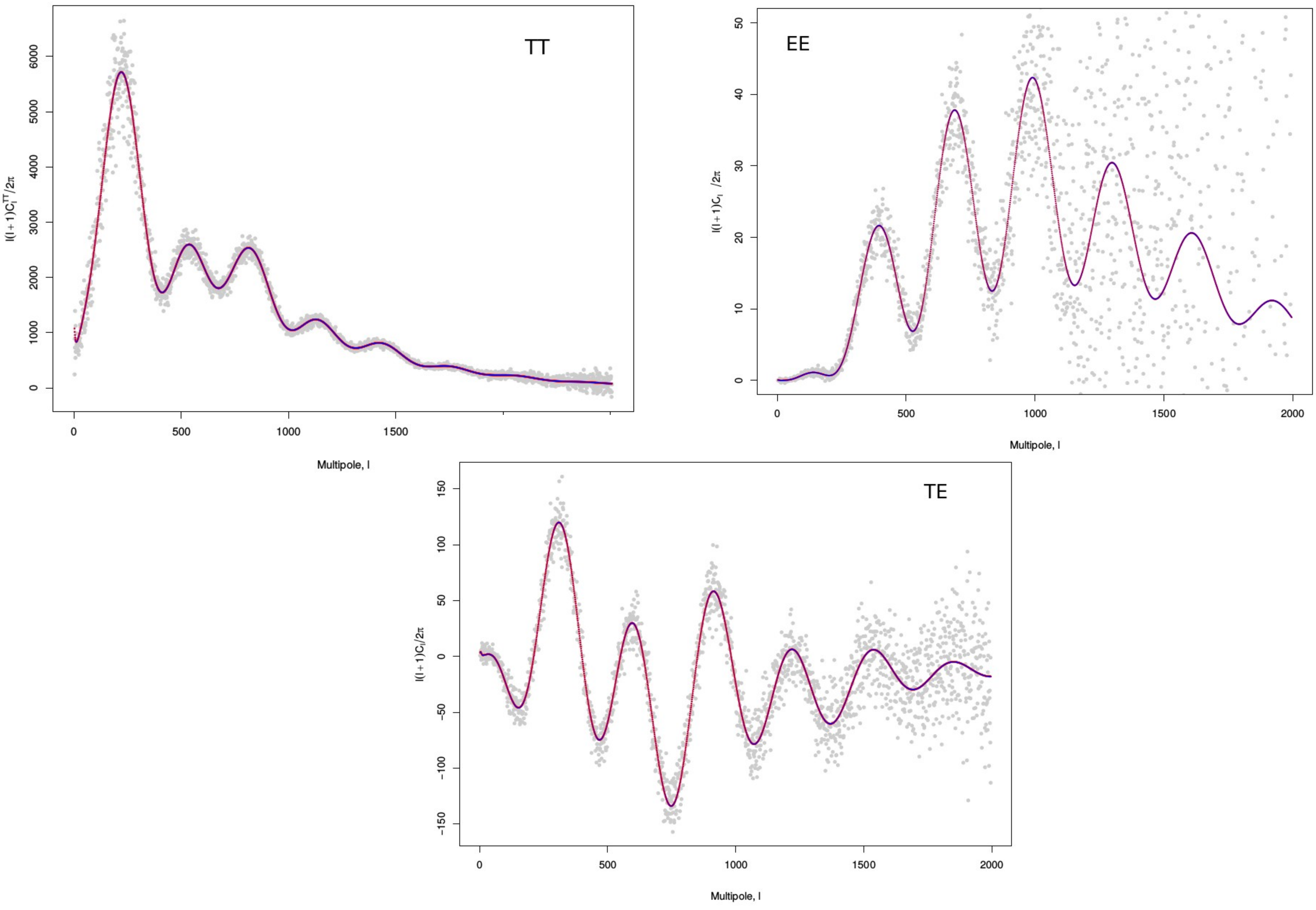
$$\mathrm{Cov}(f^*) = K(X^*,X^*) - K(X^*,X)[K(X,X)+C]^{-1}K(X,X^*)$$

$$\log(\mathfrak{L}) = -\frac{1}{2}(y-\mu)^T[K(X,X)+C]^{-1}(y-\mu) - \frac{1}{2}\log|K(X,X)+C| - \frac{n}{2}\log 2\pi$$

GP: test of LCDM model



GP: test of LCDM model



Summary

- We briefly explain the regression problem, parametric and nonparametric approach.
- REACT is a nonparametric regression technique which estimates the best fit by balancing between bias and variance.
- We apply REACT on Planck 2015 temperature and polarization angular power spectrum data.
- Comparing TT, EE and TE nonparametric fits with best-fits LCDM model shows high consistency except that some deviations in low multipoles.
- We demonstrate the harmonicity of TT angular power spectrum and estimate acoustic scale and phase shift as well as the phase difference between TT and EE angular power spectrum.
- Using Gaussian Process we estimate the TT, EE and TE angular power spectrum with zero mean function.
- We show best-fits LCDM is one of the good candidate for Planck data according to GP likelihood.

- To sum up: the best-fits LCDM model are consistent with Planck 2015 temperature and polarization angular power spectrum data using two different nonparametric tools.