Relativistic non-linear perturbation in a ΛCDM universe

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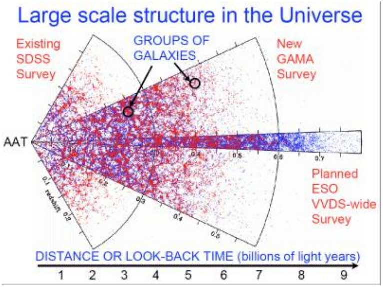
Based on J. Yoo and JG, 1602.06300 [astro-ph.CO]

Outline

Why GR in LSS?

Planned galaxy surveys: DESI, HETDEX, LSST, Euclid, WFIRST...



Larger and larger volumes, eventually accessing the scales comparable to the horizon: beyond Newtonian gravity, fully general relativistic approach (or any modification) is necessary

Why ΛCDM in non-linear regime?

- Λ (or any kind of DE) was negligible at very early times
- Λ becomes significant at later stage when non-linearities in cosmic structure are developed
- Λ affects the evolution of gravitational instability, so its effects emerge more prominently at non-linear level
- Λ is the simplest form of DE, so first to study
- No explicit analytic NL study is available yet!

What are the effects of Λ in non-linear regime of LSS?

Newtonian theory

3 basic equations for density perturbation $\delta \equiv \delta\rho/\bar{\rho}$, peculiar velocity $\boldsymbol{v}$ and gravitational potential Φ with a *pressureless* fluid

$$\dot{\delta} + \frac{1}{a}\nabla\cdot\boldsymbol{v} = -\frac{1}{a}\nabla\cdot(\delta\boldsymbol{v}) \qquad \text{continuity eq}$$

$$\dot{\boldsymbol{v}} + H\boldsymbol{v} + \frac{1}{a}\nabla\Phi = -\frac{1}{a}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \qquad \text{Euler eq}$$

$$\frac{\Delta}{a^2}\Phi = 4\pi G\bar{\rho}\delta \qquad \text{Poisson eq}$$

Newtonian system is closed at 2nd order

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = -\frac{1}{a^2}\frac{d}{dt}\left[a\nabla\cdot(\delta\boldsymbol{v})\right] + \frac{1}{a^2}\nabla\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v})$$

$\longrightarrow$ at linear order, $\delta_+ \propto a$ (growing) and $\delta_- \propto a^{-3/2}$ (decaying)

(Bernardeau et al. 2002)

Basic GR non-linear equations

Based on the ADM metric

$$ds^2 = -N^2(dx^0)^2 + \gamma_{ij}\left(N^i dx^0 + dx^i\right)\left(N^j dx^0 + dx^j\right)$$

the fully non-linear equations are (Bardeen 1980)

$$R - \overline{K}^i{}_j \overline{K}^j{}_i + \frac{2}{3}K^2 - 16\pi G E = 0$$

$$\overline{K}^j{}_{i;j} - \frac{2}{3}K_{,i} = 8\pi G J_i$$

$$\frac{K_{,0}}{N} - \frac{K_{,i}N^i}{N} + \frac{N^{;i}{}_{;i}}{N} - \overline{K}^i{}_j\overline{K}^j{}_i - \frac{1}{3}K^2 - 4\pi G(E+S) = 0$$

$$\frac{\overline{K}^i{}_{j,0}}{N} - \frac{\overline{K}^i{}_{j;k}N^k}{N} + \frac{\overline{K}_{jk}N^{i;k}}{N} - \frac{\overline{K}^i{}_k N^k{}_{;j}}{N} = K\overline{K}^i{}_j - \frac{1}{N}\left(N^{;i}{}_{;j} - \frac{\delta^i{}_j}{3}N^{;k}{}_{;k}\right) + \overline{R}^i{}_j - 8\pi G\overline{S}^i{}_j$$

$$\frac{E_{,0}}{N} - \frac{E_{,i}N^i}{N} - K\left(E + \frac{S}{3}\right) - \overline{K}^i{}_j\overline{S}^j{}_i + \frac{\left(N^2J^i\right)_{;i}}{N^2} = 0$$

$$\frac{J_{i,0}}{N} - \frac{J_{i;j}N^j}{N} - \frac{J_jN^j{}_{;i}}{N} - KJ_i + \frac{EN_{,i}}{N} + S^j{}_{i;j} + \frac{S^j{}_iN_{,j}}{N} = 0$$

Fluid quantities: $E \equiv n_\mu n_\nu T^{\mu\nu}$, $J_i \equiv -n_\mu T^\mu{}_i$, $S_{ij} \equiv T_{ij}$

Einstein-de Sitter universe

Usually, structure formation is described in EdS

$$T_{\mu\nu} = \rho_m u_\mu u_\nu \longrightarrow J_i = S_{ij} = 0$$

- Linear growth factor is all: $D_1 = a$, $D_2 = D_1^2$ and so on
- Comoving gauge ($\gamma = 0$ and $T^0{}_i = 0$) gives identical equations to the Newtonian counterparts up to 2nd order
- $N = 1$ w/o gauge freedom: coordinate time = proper time
- Pure GR contribution appears from 3rd order and is totally sub-dominant (Jeong, JG, Noh & Hwang 2011, Biern, JG & Jeong 2014)
- In e.g. synchronous gauge ($g_{00} = -1$ and $g_{0i} = 0$) we can have another Newtonian correspondence (Hwang, Noh, Jeong, JG & Biern 2015)

Linear power spectrum is obtained by solving the Boltzmann eq (e.g. `CAMB`) and is used iteratively to obtain non-linear contributions

Battle plan

Combining continuity & energy constraint eqs

$$\mathcal{H}\delta' + \frac{3}{2}\mathcal{H}^2\Omega_m\delta = \frac{a^2}{4}\left(R - \overline{K}^{ij}\overline{K}_{ij} + \frac{2}{3}\kappa^2 + 4HN^i\delta_{,i} + 4H\delta\kappa\right)$$

1. Growing solution $\delta = H\int dt\mathcal{H}^{-2}(\text{RHS})$
2. Split RHS as $\text{RHS} = \text{RHS}^{(1)} + \text{RHS}^{(2)} + \text{RHS}^{(3)} + \cdots$ with

$$\text{RHS}^{(n)}(t,\boldsymbol{x}) \equiv \sum_I \text{RHS}_I^{(n)}(t,\boldsymbol{x}) = \sum_I X_{mI}^{(n)}(\boldsymbol{x})T_{mI}(t)$$

[n: n-th order, I: t-dep, $m(\leq n)$: growth factor $\propto D_1^m$ in EdS]

3. With $\delta = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + \cdots$ each analytic solution is given by

$$\delta^{(n)} = \sum_I \delta_{mI}^{(n)}(t,\boldsymbol{x}) = \sum_I D_{mI}(t)X_{mI}^{(n)}(\boldsymbol{x}) \quad \text{with} \quad D_{mI}(t) = H\int dt\frac{T_{mI}}{\mathcal{H}^2}$$

Linear order solution

At linear order

$$\mathrm{RHS}^{(1)} = -\Delta\varphi^{(1)}(\boldsymbol{x}) \equiv X_1^{(1)}(\boldsymbol{x}) \quad \text{and} \quad T_1(t) = 1$$

Thus we recover the well-known linear solution, with $\varphi^{(1)} \equiv \mathscr{R}$

$$\delta_1^{(1)}(t,\boldsymbol{x}) = D_1(t) X_1^{(1)}(\boldsymbol{x}) \quad \text{with} \quad D_1(t) \equiv H \int \frac{dt}{\mathscr{H}^2}$$

we can further define $f_1 \equiv \dfrac{d\log D_1}{d\log a}$ and $\Sigma_1 \equiv 1 + \dfrac{3}{2}\dfrac{\Omega_m}{f_1}$

Second order solutions

Likewise at 2nd order

$$\text{RHS}^{(2)} \sim \left(\text{constant}, \frac{1}{\mathcal{H}^2 f_1 \Sigma_1}, \frac{1}{\mathcal{H}^2 \Sigma_1^2}, \frac{1}{\mathcal{H}^2 f_1 \Sigma_1^2} \right)$$

Thus other than D_1 (coming from const RHS) 3 new growth factors

$$D_{2A} = \frac{7}{5} \int dt D_1^2 f_1 \Sigma_1 \,, \quad D_{2B} = \frac{7}{2} H \int dt D_1^2 f_1^2 \,, \quad D_{3C} = \frac{7}{2} H \int dt D_1^2 f_1$$

Not all D_{2I}'s are indep but $D_{2A} + D_{2C} = 2D_1^2$, so we can write the pure Newtonian 2nd order solution explicitly

$$\delta^{(2)}(t, \boldsymbol{x}) = \delta_1^{(2)} + \underbrace{\sum_{I=A}^{C} \delta_{2I}^{(2)}}_{\equiv \delta_2^{(2)}} = D_1 X_1^{(2)} + \sum_{I=A}^{C} D_{2I} X_{2I}^{(2)} \quad \text{with}$$

$$\delta_2^{(2)} = \frac{5D_{2A} + D_{2B} + 4D_{2C}}{10} \left[\frac{5}{7} \left(\mathscr{R}^{,i} \Delta \mathscr{R} \right)_{,i} \right] + \frac{5D_{2A} - D_{2B}}{4} \left[\frac{\Delta}{7} \left(\mathscr{R}^{,i} \mathscr{R}_{,i} \right) \right]$$

3rd order solutions (1/2)

At 3rd order, RHS has various time dependences: e.g. $\varphi^{(3)}$ reads

$$\varphi^{(3)} \sim \left(\text{constant}, D_1, D_1^2, D_{2I}\right)$$

Accordingly we have components proportional to D_1 and D_{2I}:

$$\delta^{(3)} \supset \underbrace{\delta_1^{(3)}}_{\propto D_1} + \underbrace{\delta_{2A}^{(3)}}_{\propto D_{2A}} + \underbrace{\delta_{2B}^{(3)}}_{\propto D_{2B}} + \underbrace{\delta_{2C}^{(3)}}_{\propto D_{2C}}$$

3rd order solutions (2/2)

And new growth factors that all scale as D_1^3 in EdS:

$$D_{3D} = \frac{9}{5} H \int dt D_1^3 f_1 \Sigma_1 \quad \text{with} \quad X_{3D}^{(3)} = \text{too long!} \quad (1)$$

$$D_{3E} = \frac{9}{2} H \int dt D_1^3 f_1 \quad \text{with} \quad X_{3E}^{(3)} = \text{too long!} \quad (2)$$

$$D_{3F} = \frac{9}{2} H \int dt D_1^3 f_1^2 \quad \text{with} \quad X_{3F}^{(3)} = \text{too long!} \quad (3)$$

and those coming from $\delta_{2I}^{(2)}$ that also scale as D_1^3 in EdS:

$$D_{3I} = \frac{9}{5} H \int dt D_1 f_1 \Sigma_1 D_{2I} \quad \text{with} \quad X_{3I}^{(3)} = -\frac{5}{18} \left[\left(\mathscr{R}^{,ij} \Delta^{-1} \partial_j + \Delta \mathscr{R} \Delta^{-1} \partial^i \right) X_{2I}^{(2)} \right]_{,i}$$

$$D_{3I'} = \frac{9}{4} H \int dt D_1 D_{2I} f_{2I} \quad \text{with} \quad X_{3I'}^{(3)} = -\frac{4}{9} \left(\Delta \mathscr{R} \Delta^{-1} X_{2I}^{(2),i} \right)_{,i}$$

$$D_{3I''} = \frac{9}{2} H \int dt D_1 f_1 D_{2I} \quad \text{with} \quad X_{3I''}^{(3)} = -\frac{2}{9} \left(X_{2I}^{(2)} \mathscr{R}^{,i} \right)_{,i}$$

$$D_{3I'''} = \frac{9}{4} H \int dt D_1 f_1 D_{2I} f_{2I} \quad \text{with} \quad X_{3I'''}^{(3)} = \frac{2}{9} \left(\mathscr{R}^{,ij} \Delta^{-1} \partial_i \partial_j - \Delta \mathscr{R} \right) X_{2I}^{(2)}$$

Previous GR solutions

1-loop power/bi-spectrum of δ (Jeong et al. 2011, Biern et al. 2014)

1. Initial condition at $t = t_i$ is set by δ rather than φ
2. Linear initial condition: $\delta(t_i) = \delta_1^{(1)}(t_i)$

t-dep \ pert order	1st	2nd	3rd
$\sim D_1$ in EdS	$\delta_1^{(1)}$	$\delta_1^{(2)}$	$\delta_1^{(3)}$
$\sim D_1^2$ in EdS		$\delta_2^{(2)}$	$\delta_2^{(3)}$
$\sim D_1^3$ in EdS			$\delta_3^{(3)}$

$\delta(t,\boldsymbol{x}) = \dfrac{c(\boldsymbol{x})}{\mathcal{H}^2} + \cdots$ with assuming linear energy constraint

$$c(\boldsymbol{x}) = -\frac{2}{5}\Delta\mathscr{R}$$

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$\sim D_1^3$ in EdS			$\delta_3^{(3)}$

$\delta(t,\boldsymbol{x}) = \dfrac{c(\boldsymbol{x})}{\mathcal{H}^2} + \cdots$ with**out** assuming **non-**linear energy constraint

$$c(\boldsymbol{x}) = -\frac{2}{5}\Delta\mathscr{R} + \frac{2}{5}\left[\frac{3}{2}\mathscr{R}^{,i}\mathscr{R}_{,i} + 4\mathscr{R}\Delta\mathscr{R} - 3\mathscr{R}\left(3\mathscr{R}^{,i}\mathscr{R}_{,i} + 4\mathscr{R}\Delta\mathscr{R}\right)\right] + \cdots$$

Previous Newtonian solutions

Upon identifying $\delta \to \delta_N$ and

$$-\kappa \to \frac{1}{a}\nabla\cdot\boldsymbol{v}_N \equiv \theta_N$$

energy conservation and trace ADM equations become identitcal to the Newtonian continuity and Euler equations

$$\delta_N(t,\boldsymbol{k}) = \sum_{n=1}^{\infty} D^n(t)\delta^{(n)}(\boldsymbol{k})$$
$$\frac{\theta_N(t,\boldsymbol{k})}{Hf_1} = \sum_{n=1}^{\infty} D^n(t)\theta^{(n)}(\boldsymbol{k})$$

with the initial condition $\delta_N(t_i,\boldsymbol{k}) = \delta_1^{(1)}(t_i,\boldsymbol{k}) \equiv \hat{\delta}(\boldsymbol{k})$ $[D_1(t_i) = 1]$

Conclusions

- As galaxy surveys become deeper and deeper, fully GR description is relevant
- With a non-zero cosmological constant Λ:
  - Proper-time hypersurface provides Newtonian intuition
  - Perturbative analytic solutions can be obtained
  - Initial non-linearity in δ in terms of $\mathscr{R}$
- Directly connected to inflation